Linear Current Flow Equations with Application to Distribution Systems Reconfiguration

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***Abstract*—Conventionally, power flow equations are used for distribution systems (DS) analysis to find the nodal voltages. For the particular form of the DS reconfiguration problem, however, a direct formulation in terms of branch flows allows a substantial increase in solution efficiency from an optimization point of view. In this paper, a set of linear current flow (LCF) equations are derived for DS. This formulation is then used within the network reconfiguration problem for loss minimization. A mixed-integer quadratically constrained programming (MIQCP) formulation, together with a mixed-integer linear programming (MILP) formulation are proposed in this paper and assessed through simulations. In these comparisons, the MILP formulation shows computational advantages over the MIQCP version and the preceding literature. The proposed methods are evaluated on several test systems.**

***Index Terms*—Distribution systems, linear current flow equations, voltage-dependent load model, network reconfiguration.**

I. Introduction

THE tendency towards optimizing the utilization of the current infrastructure in power systems has significantly increased in the past decade. The costs and technical difficulties associated with building new lines has been a major motivation [1]. Also, reducing power losses has become a desirable objective for many distribution companies (DISCOs). The problem of distribution system (DS) reconfiguration for purposes such as loss minimization, load balancing, and voltage profile improvement has been widely studied since the early 80's. Direct mathematical formulations, as well as heuristic methods have been proposed to solve these problems. In direct mathematical methods, the optimization problem is usually formulated as a mixed-integer programming (MIP) problem and it is then solved using commercial MIP solvers. The advantages of these methods can be summarized as follows:

- Flexibility for adding new constraints or optimizing for different objectives
- Provide the optimality gap at each iteration
- Guarantee robustness and reliability

While the MIP-based methods have the mentioned advantages, they have a significant drawback: high computational burden. This drawback impedes the application of the MIP-based methods for large-scale systems. Here is where the heuristic methods step forward. The advantage of heuristic algorithms over direct mathematical approaches lies within their reduced computational complexity. However, this advantage comes at the cost of finding suboptimal solutions. Also, there is usually no information on the optimality level (optimality gap) of the solution and the robustness of the algorithm may not be guaranteed.

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A. Motivation

The above discussion reveals the important fact that, when attempting to find a good optimization algorithm, one needs to take advantage of the low computational burden of heuristic methods and the flexibility and reliability of the direct mathematical methods, simultaneously. In the MIP formulation of the problem, the basic challenge is the nonlinear power flow equations. Using the conventional power flow formulations, the resulting MIP problem would belong to the family of mixed-integer nonlinear programming (MINLP) problems. The large-scale MINLP problems are extremely difficult to solve. Therefore, researchers have tried to come up with different versions of power flow formulations with simplifying assumptions to avoid the non-convex nonlinear equations. In addition, the power flow equations attempt to find the nodal voltages instead of directly calculating the power flows. The present paper provides a linear current flow (LCF) formulation that suits the structure of the reconfiguration problem. This formulation is based on the voltage-dependent load model introduced by the authors in [2].

The recent advances in MIP algorithms have led to solvers capable of handling mixed-integer quadratically constrained programming (MIQCP) problems. However, these solvers are slow when compared to the mixed-integer linear programming (MILP) ones if many conic quadratic constraints are present. This motivated the authors to use efficient relaxation techniques in order to provide a MILP formulation of the problem. The computational advantages of the MILP formulation is shown against the MIQCP version.

B. Related Work

1) Direct Mathematical Methods: Three convex models for the distribution system reconfiguration problem are derived in [3]. In this work, the *DistFlow* equations, originally proposed in [4], are used. The first method is a quadratic programming formulation obtained by fixing all the nodal voltages to 1.0 in per-unit. The second formulation is a quadratically-constrained programming algorithm that takes into account the branch ampacities. The third formulation is a second-order cone programming algorithm that is obtained by approximating the nonlinear *DistFlow* equations. The last approximation method is the most accurate but requires prohibitive computation.

A mixed-integer conic programming reformulation of the reconfiguration problem is proposed in [5]. The nonlinear power flow equations are rewritten in terms of rotated conic quadratic constraints. A MILP formulation is also derived by replacing the conic quadratic constraints with their linear approximations. Numerical solutions indicate that both methods struggle to provide an optimal solution in a reasonable time. Suboptimal solutions (assuming 5% optimality gap), however, are obtained within a relatively short time.

Reference [6] applies a decomposition algorithm to the MINLP formulation of the reconfiguration problem. The proposed algorithm is based on Benders decomposition, which solves the problem in two stages, namely the *master* and *subproblem*. The application of Benders decomposition to non-convex MINLP problems is just a heuristic since the original method, known as generalized Benders decomposition, assumes convex nonlinear functions [7] when fixing the binary variables, while the power flow equations are not convex. Also, the master problem is formulated as a MINLP problem, which is still a barrier since it is difficult to solve this problem for large-scale systems.

A MINLP formulation of the reconfiguration problem is proposed in [8] and is solved using a branch-and-bound algorithm. Although this formulation has only bilinear terms as its nonlinear constraints, it is still a non-convex quadratic model, which is difficult to solve.

A MILP formulation for the reconfiguration problem is proposed in [9]. The power flow equations are written in the form of current injections and the nonlinear terms are linearized using a least-square method at each node. The first drawback of this method is that for every single node, a linear approximation should be calculated. Another drawback is that this formulation takes the nodal voltages as the independent variables, which creates a complicated structure for the resulting MILP problem. These drawbacks are addressed in the present paper.

The present authors proposed a framework for distribution systems optimization using a linear power flow (LPF) formulation in [10]. The application of the LPF in the network reconfiguration problem has shown computational advantages over the related literature. However, due to the nature of the LPF formulation, multiplications of the binary and continuous variables are present in the model. This difficulty has been dealt with by introducing auxiliary variables and constraints, which reduced the potential speed up that could be achieved by the linear formulation. A reformulation of the LPF in terms of current flows, i.e. the linear current flow (LCF), is preferable in the case of network reconfiguration.

2) Heuristic Methods: Numerous heuristic methods have been proposed in the literature for the problem of DS reconfiguration. These methods can be categorized into two main groups. In the first group, knowledge of power systems engineering is applied to building a heuristic method, without using artificial intelligence elements. Good examples of this type of heuristics are given in [4], [11]-[17]. Normally, these methods do not guarantee the optimality of the results and, in most cases, there is no knowledge of the optimality gap. Nonetheless, these methods usually provide good-quality solutions within a short time and show a robust performance.

In the second group of heuristic methods, the problem is solved using a variety of artificial intelligence techniques. Some good examples of these methods are Genetic Algorithms [18], Plant Growth Algorithms [19], Tabu Search Algorithms [20], Harmony Search Algorithms [21], Evolutionary Algorithms [22], Artificial Neural Networks [23], Simulated Annealing Algorithms [24], Ant Colony Algorithms [25], and Particle Swarm Optimization [26]. This type of heuristic methods generally admit fast solutions. However, the optimality of the results is not usually guaranteed. Also, the robustness of these reconfiguration algorithms relies highly on the robustness of the applied artificial intelligence method.

C. Contributions

The mixed-integer programming formulations proposed in the preceding literature share a common structure, which is to take the nodal voltages as independent variables. On the other hand, due to the nature of the reconfiguration problem, the main driving variables are the current flows. In fact, the binary variables representing the status of a switch are in direct control of the the current flow of each branch. Using current flows instead of nodal voltages as the main variables gives such a structure to the resulting combinatorial optimization problem, which can be solved more efficiently by the standard branch-and-cut routines.

The main contributions of this work are twofolds. The LCF equations are derived here, which can be used to calculate the current flows directly, independent of voltage values. The LCF equations are then used to formulate the network reconfiguration problem. Using the LCF-based MILP formulation, it is possible to reduce the problem dimension by dropping the voltage variables and constraints in the minimum-loss network reconfiguration problem, which is a valid assumption for some systems [12]. Effective relaxation techniques are then applied to replace the quadratic constraints with their polyhedral relaxations. It is shown, through extensive simulations, that the proposed formulation results in reduced computation times, while providing a proven optimal solution.

II. Distribution System Modeling

A. Load Modeling

The constant P-Q load model, as well as the common voltage-dependent load models (e.g., ZIP and exponential models) introduce nonlinearity in the solution of the power flow equations. The load model proposed in [2] is an alternative to the conventional voltage-dependent load models that allows for a linear formulation of the PF equations. The proposed load model is described as:

$$\frac{P(V)}{P_0} = C_Z \left(\frac{V}{V_0}\right)^2 + C_I \left(\frac{V}{V_0}\right) \tag{1}$$

$$\frac{Q(V)}{Q_0} = C'_Z \left(\frac{V}{V_0}\right)^2 + C'_I \left(\frac{V}{V_0}\right) \tag{2}$$

in which P_0 and Q_0 are the active and reactive power demand at V_0; constants C and C' are calculated by a curve-fitting procedure applied to the load measured data. Note that there is only one independent parameter in (1) and (2) because $C_Z + C_I = 1$ and $C'_Z + C'_I = 1$ [2].

B. The Linear Current Flow Equations

The linear power flow (LPF) equations derived in [2] are based on the Norton equivalent of the load synthesis, as shown in Fig. 1. The equivalent current source and impedance, in complex numbers, are calculated as:

$$Y_{Li} = \frac{P_0 C_Z}{V_0^2} - j\frac{Q_0 C'_Z}{V_0^2} \tag{3a}$$

$$I_{Li} = \frac{P_0 C_I}{V_0} - j\frac{Q_0 C'_I}{V_0} \tag{3b}$$

Using nodal analysis and the mentioned load model, a linear formulation was obtained in [2] that calculates the nodal voltages. To find the branch currents, extra computations are required.

In a different approach, let us consider the branch currents, instead of the nodal voltages, as the independent variables. Also, replace the Norton equivalent of the loads shown in Fig. 1 by their Thévenin equivalent, as shown in Fig. 2. Next, assume an arbitrary current direction for each branch, as is done in Fig. 2. Considering the loop formed through the ground by the load connected to Node i, Line k, and the load connected to Node j, we can apply Kirchhoff's Voltage Law to obtain:

$$\begin{aligned} -E_i + Z_{Li}(-I_k + \sum_{l\in N_i^{\text{in}}} I_l - \sum_{l\in N_i^{\text{out}}} I_l) - Z_k I_k \\ - Z_{Lj}(I_k + \sum_{h\in N_j^{\text{in}}} I_h - \sum_{h\in N_j^{\text{out}}} I_h) + E_j = 0 \end{aligned} \tag{4}$$

In (4), $N_i^{\text{in}}/N_i^{\text{out}}$ are the sets of branches entering/leaving Node i excluding Branch k. Rearranging this equation, one can write for Branch k,

$$\begin{aligned} (Z_{Li} + Z_{Lj} + Z_k)I_k - Z_{Li}\sum_{l\in N_i^{\text{in}}} I_l + Z_{Li}\sum_{l\in N_i^{\text{out}}} I_l \\ + Z_{Lj}\sum_{h\in N_j^{\text{in}}} I_h - Z_{Lj}\sum_{h\in N_j^{\text{out}}} I_h) = E_j - E_i \end{aligned} \tag{5}$$

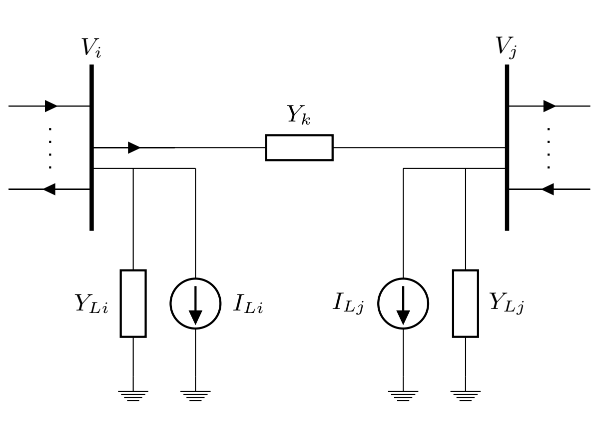


Figure 1. A generic part of distribution system: Norton equivalent.

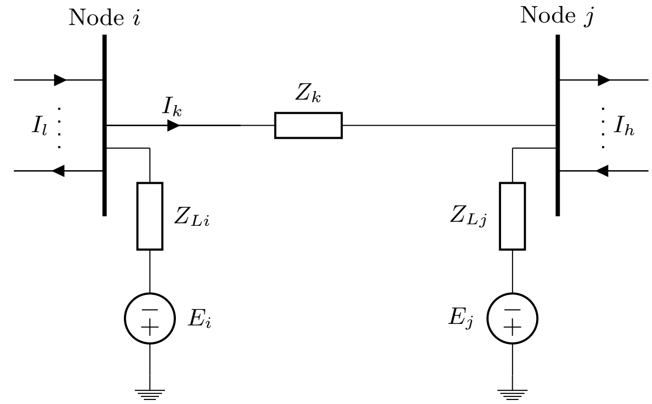


Figure 2. A generic part of distribution system: Thévenin equivalent.

The linear current flow (LCF) equations can now be written as:

$$Z\,I = E \tag{6}$$

in which $Z \in \mathbb{C}^{m\times m}$ is the network impedance matrix (m is the total number of branches); $I \in \mathbb{C}^{m\times 1}$ is the vector of branch currents; $E \in \mathbb{C}^{m\times 1}$ is the vector of equivalent voltage source for each branch. The k^{th} element of E is the difference between the Thévenin voltage sources located at both ends of Branch k, as in (5). The non-zero elements on the k^{th} row of Z, assuming Branch k is connected between Nodes i and j, can be defined as:

$$z_{k,l} = \begin{cases} Z_{Li} + Z_{Lj} + Z_k & l = k \\ -Z_{Li} & l \in N_i^{\text{in}} \\ Z_{Li} & l \in N_i^{\text{out}} \\ Z_{Lj} & l \in N_j^{\text{in}} \\ -Z_{Lj} & l \in N_j^{\text{out}} \end{cases} \tag{7}$$

The definition in (7) is readily implemented in a computerized program to form Z for a generic distribution system. A noticeable feature of the Z matrix is that, unlike the admittance matrix, it is not necessarily symmetrical.

Many optimization software packages only allow for operations on real numbers. Therefore, it is worthwhile to derive the current flow equations (6) in rectangular (Cartesian) coordinates. Assume $Z = R + jX$, $I = I^{\text{re}} + jI^{\text{im}}$, and $E = E^{\text{re}} + E^{\text{im}}$. It is trivial to verify the following variant of (6):

$$\begin{bmatrix} R & -X \\ X & R \end{bmatrix} \begin{bmatrix} I^{\text{re}} \\ I^{\text{im}} \end{bmatrix} = \begin{bmatrix} E^{\text{re}} \\ E^{\text{im}} \end{bmatrix} \tag{8}$$

which has $2m$ variables and $2m$ equations. Using the LCF equations of (6) or (8), the branch currents can be directly calculated, independently from the nodal voltages.

III. A LCF-Based MIQCP Formulation of the Network Reconfiguration Problem

The LCF equations derived in Section II are used here to formulate the optimal network reconfiguration problem. From a mathematical point of view, the LCF formulation suits the reconfiguration problem better than the LPF variant. This is mainly to avoid the bilinear terms resulting from multiplication of binary and continuous variables in [10]. With LCF, the disconnection of each line is simply modeled by forcing its current to zero, while with LPF the corresponding line impedance has to be taken out from the impedance matrix.

The objective of the problem can be load balancing among feeders, loss reduction, voltage profile improvement, etc. Here, the loss minimization problem is studied. The total losses in the network can be calculated as:

$$P_{\text{loss}} = \sum_{k=1}^{m} R_k(|I_k^{\text{re}}|^2 + |I_k^{\text{im}}|^2) \tag{9}$$

The constraints to the minimum-loss network reconfiguration are established next.

A. The Current Flow Equations

The LCF equations in (8) can be written in expanded form as

$$\sum_{k=1}^{m} \left(R_k I_k^{\text{re}} - X_k I_k^{\text{im}}\right) = E_k^{\text{re}} \tag{10a}$$

$$\sum_{k=1}^{m} \left(X_k I_k^{\text{re}} + R_k I_k^{\text{im}}\right) = E_k^{\text{im}} \tag{10b}$$

Note that (10a) and (10b) add $2 \times m$ linear equality constraints to the optimization problem. This system of equations has a single unique solution. In other words, the matrix representing the system of linear equations has full rank. When a few of the branches are disconnected to retrieve a radial structure, the corresponding equations to those branches have to be eliminated. Otherwise, the system of linear equations is overdetermined and, thus, infeasible. The process of branches disconnection is carried out as follows.

Let us denote the status of branch k by u_k. It is then desired to activate the k^{th} equation in (10a) and (10b) if $k = 1$, and deactivate it otherwise. This particular type of variables, i.e. u_k, are called *indicator variables* [27]. An indicator variable, when is zero, forces some of the problem variables to assume a fixed value, and, otherwise, forces them to belong to a convex set. Commercial mixed-integer programming solvers such as CPLEX are capable of efficiently handling this feature of the problem [28].

An indicator variable can also be represented as a disjunction [29]. Disjunctive programming deals with logic-based constraints, i.e. constraints that are only active when an indicator variable is *true*. One reformulation of a disjunction, e.g., (10), takes the following form:

$$(u_k - 1)M \leq \sum_{k=1}^{m}(R_k I_k^{\text{re}} - X_k I_k^{\text{im}}) - E_k^{\text{re}} \leq (1 - u_k)M \tag{11a}$$

$$(u_k - 1)M \leq \sum_{k=1}^{m}(X_k I_k^{\text{re}} + R_k I_k^{\text{im}}) - E_k^{\text{im}} \leq (1 - u_k)M \tag{11b}$$

in which M is a large-enough positive scalar. This reformulation is also known as Big-M formulation. It is tricky to find a good value for M since a large value leads to loose bounds and a small value may lead to an infeasible problem. Commercial solvers such as CPLEX are capable of finding good values for M during the solution process to tighten the bounds. This useful feature of the solver is utilized here to enforce the disjunctions.

B. Branch Ampacities

The magnitude of the current flowing through a branch is bounded by the thermal limit of the conductors. This constraint can be represented as:

$$|I_k^{\text{re}}|^2 + |I_k^{\text{im}}|^2 \leq |I_k^{\max}|^2 \, u_k \tag{12}$$

where $I^{\max}$ is the maximum current magnitude allowed in a branch. Note that $u_k = 0$ forces both I_k^{re} and I_k^{im} to be zero.

C. Nodal Voltage Limits

The voltage at each node has to be within certain operational limits, e.g., $\pm 5\%$. In order to derive the voltages from the branch currents, the Thévenin equivalent of loads shown in Fig. 2 is used. The voltage at Node i can be calculated as

$$V_i^{\text{re}} + jV_i^{\text{im}} = -E_i + Z_{Li}(-I_k + \sum_{l \in N_i^{\text{in}}} I_l - \sum_{l \in N_i^{\text{out}}} I_l) \tag{13}$$

This equation is in the form of complex numbers. The real and imaginary parts of nodal voltages, which are linear combinations of currents, can be used to impose the voltage magnitude limits:

$$V_i^{\min} \leq \sqrt{|V_i^{\text{re}}|^2 + |V_i^{\text{im}}|^2} \leq V_i^{\max} \tag{14}$$

Recall the assumption of small voltage angles in DS [2]. This assumption allows for eliminating the term $|V_m^{\text{im}}|^2$ in (14), giving the following box constraint [10]:

$$V_i^{\min} \leq V_i^{\text{re}} \leq V_i^{\max} \tag{15}$$

where V_i^{re} can be retrieved from (13) as a function of currents.

D. Radiality Constraint

When radiality of the final configuration is imposed, the formulation proposed in [5] is used. In this formulation, the network is converted into a directional graph by assigning two binary variables to each branch. The aim is to force each node to have exactly one *parent*.

IV. A MILP Formulation of the Network Reconfiguration Problem

There are two quadratic terms in the MIQCP formulation, i.e. (9) and (12). In this section, effective relaxation techniques are applied to the quadratic terms to find a MILP formulation for the problem.

A. Active Power Losses

The total active power losses are calculated in (9). Each element of the summation, i.e. $R_k(|I_k^{\text{re}}|^2+|I_k^{\text{im}}|^2)$, is a convex quadratic term. A piecewise linearization (PWL) technique is proposed in [30] to linearize these type of functions. This technique has been proven efficient in many applications, e.g., in linearizing the generators cost functions [31]. The details of this linearization process are provided in Appendix A. When applying the PWL technique, each term in the objective function takes the form of a maximum over a set of affine functions. Define λ_k as

$$\lambda_k = \max_{1\leq i\leq s}\{\alpha_i I_k^{\text{re}} + \beta_i I_k^{\text{im}} + \gamma_i\} \tag{16}$$

which is the PWL approximation of $|I_k^{\text{re}}|^2 + |I_k^{\text{im}}|^2$. The objective function then takes the following form:

$$\min \quad \sum_{k=1}^{m} R_k \max_{1\leq i\leq s}\{\alpha_i I_k^{\text{re}} + \beta_i I_k^{\text{im}} + \gamma_i\} \tag{17}$$

which is a min-max optimization problem. This can be easily reformulated as a linear programming problem, as discussed in [32]. The following problem is equivalent to (17):

$$\min \quad \sum_{k=1}^{m} R_k \lambda_k \tag{18a}$$

$$\text{s.t.} \quad \lambda_k \geq \alpha_i I_k^{\text{re}} + \beta_i I_k^{\text{im}} + \gamma_i, \quad \forall i = 1,\ldots,s. \tag{18b}$$

The number of linear pieces, i.e. s, can be chosen in a way to satisfy a particular accuracy in the approximation.

B. Branch Ampacities

The current ampacity constraint in (12), for a fixed $u_k = 1$, is a quadratic constraint of the form (23). The branch status u_k is enforced later by adding (19) to the problem. In Appendix B, a relaxation method for these constraints is proposed. Using a hexagon relaxation (i.e. $n = 6$), the linear constraints of (28) replace (12). In addition, the following two constraints are required enforce the branch status:

$$-u_k I_k^{\max} \leq I_k^{\text{re}} \leq u_k I_k^{\max} \tag{19a}$$

$$-u_k I_k^{\max} \leq I_k^{\text{im}} \leq u_k I_k^{\max} \tag{19b}$$

Theses equations would be inactive if $u_k = 1$, since they represent a larger feasible region that contains the hexagon defined by (28).

V. Simulation Results

In this section, the proposed MIQCP and MILP formulations are applied to several distribution test systems. The commonly used test systems in the literature are the 16-node [11], 33-node [4], 70-node [33], 119-node [20], 136-node [34], 415-node [5], and 880-node [35]. In the simulations, the load models described in Section II-A are used and the parameters for all the loads, without loss of generality, are assumed to be identical as $C_Z = C_I = 0.5$. The value of losses obtained using the MIQCP and MILP methods are shown in Table I. In this table, the losses are calculated using constant P-Q load models and conventional power flow, with the configuration obtained using the proposed methods, to be able to compare the results with the previous literature. The linearization technique used to represent the objective function in the MILP formulation, i.e. (18), introduces a negligible error in the value of losses. It should be noted that the direction at which the losses increase/decrease plays a more important role in the optimization problem than its exact value. Due to this reason, both the MILP and MIQCP methods find the same optimal configurations. The optimal configurations are given in Table II. The open switches (tie switches) are reported in this table.

Optimal reconfiguration for loss minimization invariably improves the voltage profile of the network. This fact has been observed by the authors in all the test systems, and the observations in [12] also support this fact. Therefore, it may be possible to drop the voltage limit constraints, i.e. (15), in most of the cases when minimizing the losses. The proposed formulation provides such a structure that allows for completely dropping the voltage variables and constraints, without affecting the results. The problem dimension would then reduce substantially. This fact further supports the suitability of the proposed LCF-based formulation for network reconfiguration.

The computation times for the test systems are given in Table III. All the simulations were done in the AIMMS platform [36] on an Intel Core i7-2600, 3.4 GHz CPU, 8 GB RAM, and 64-bit operating system machine. The MIP solver, i.e. CPLEX, allows the user to provide a known initial solution for the problem. In this paper, a fast heuristic method developed by the authors is employed to initialize the solution process. This heuristic method finds the minimum spanning tree for the graph representing the network with all the switches closed and the current magnitudes taken as its weights. This method is more efficient than the general-purpose heuristics used by CPLEX to find an initial solution.

It is important to notice the difference between the computation time for MILP and MIQCP formulations. Although the size of the MILP problem is larger than the MIQCP version, the computational complexity of the linear programming problem solved at each iteration is less than the equivalent conic quadratic programming problem. The main challenge in the MIQCP problem is to handle the conic constraints. In other words, if the conic constraints are relaxed, then all the constraints are linear and only the objective is quadratic. Linearizing the quadratic objective only makes a small difference, as compared to relaxing the conic constraints, in the solution time. When the control variables are only discrete variables, which is the case in network reconfiguration, using linearized objective function is well-justifiable, as is also done in [37] (the first stage of the 2-Stage Procedure) for the unit commitment problem.

VI. Conclusion

The linear current flow (LCF) equations derived in this paper are advantageous in problems that directly deal with current flows. One problem that needs direct calculation of the

Table I
SIMULATION RESULTS FOR SYSTEM RECONFIGURATION: OPTIMAL SOLUTIONS

#Nodes	14	33	70	119	136	415	880
P_{loss}(kW) by MIQCP/MILP	468.3	139.5	201.4	869.7	280.2	2352.4	991.4
Reference	[11]	[3]	[33]	[20]	[5]	[5]	[3]
P_{loss}(kW)	468.3	139.5	205.1	869.7	280.2	2359.9	999.1

Table II
OPTIMAL CONFIGURATIONS FOR TEST SYSTEMS

#Nodes	Open Lines
14	7,8,16
33	7,9,14,32,37
70	14,30,45,51,66,70,75,76,77,78,79
119	23,26,34,39,42,51,58,71,74,95,97,109,122,129,130
136	7,35,51,90,96,106,118,126,135,137,138,141,142,144, 145,146,147,148,150,151,155
415	7,33,39,42,63,72,82,84,86,88,89,90,92,103,129,135,138,159, 168,178,180,182,184,185,186,188,199,225,231,234,255,264, 274,276,278,280,281,282,284,295,321,327,330,351,360,370, 372,374,376,377,378,380,391,417,423,426,447,456,466,468, 470,472,473,474,476,481,482,483,484,485,486,487,488
880	84,131,140,159,190,244,282,288,306,312,409,411,452,494, 596,616,629,631,637,698,815,844,885,888,889,890,900

currents is the network reconfiguration problem. The structure of the problem can be built in such a way that forcing a current to zero can be readily achieved with the LCF equations. As compared to similar studies, the proposed formulation exhibits better performance in terms of computation times. The relaxation techniques applied show a strong impact on the computations. Other areas of application for the proposed LCF equations are network reconfiguration for reliability enhancement and system restoration. For meshed distribution systems, performing $N-1$ contingency analysis for branch outages is directly possible using the LCF formulation.

APPENDIX A
PIECEWISE LINEARIZATION OF A QUADRATIC FUNCTION

Consider the following standard quadratic function:

$$f(x,y) = x^2 + y^2 \tag{20}$$

where x and y are bounded by box constraints $x_l \leq x \leq x_u$ and $y_l \leq y \leq y_u$. The max-affine piecewise linearization (PWL) of $f(x,y)$ can be expressed as:

$$\hat{f}(x,y) = \max_{1\leq i\leq s} \{\alpha_i x + \beta_i y + \gamma_i\} \tag{21}$$

which has no restrictions on the subspaces over which the affine pieces are defined. The best $\hat{f}(x,y)$, with a fixed s, can then be obtained by solving the following least-squares problem:

$$\min_{\alpha_i,\beta_i,\gamma_i} \sum_{k=1}^{w} \left|\hat{f}(x_k,y_k) - f(x_k,y_k)\right|^2 \tag{22}$$

where w is the number of point-wise function evaluations. The superiority of this PWL technique over other available methods is that it decides on the subspaces over which the affine pieces are defined [31]. An efficient method is proposed in [30] to find the solution of (22). In addition, there are commercial solvers capable of handling these types of problems involving the "max" operator, which are usually categorized as *non-smooth* problems. Good examples are CONOPT, MINOS, LGO and IPOPT.

The standard quadratic function in (20) is shown in Fig. 3(a), as an example, for $x, y \in [-4, 4]$. The PWL approximation of this function using 24 pieces ($s = 24$) is shown in Fig. 3(b).

Table III
COMPUTATION TIMES (S)

	Test System (# Nodes)					
	33	**70**	**119**	**136**	**415**	**880****
MIQCP	35.2	485	1009	1785	1800+	1800+
MILP	0.15	0.91	2.8	5.6	116	398
[10]	3.2	5.7	39.4	188.4	953	1134
[3]	12.8	11310	N/A	2.35*	N/A	3192*
[5]	N/A	N/A	N/A	1800+	1800+	1800+
[38]	19	N/A	4007	4473	14256	N/A

* Suboptimal solutions obtained by intense relaxations [3].
** Relative optimality gap is considered 1% for this case.

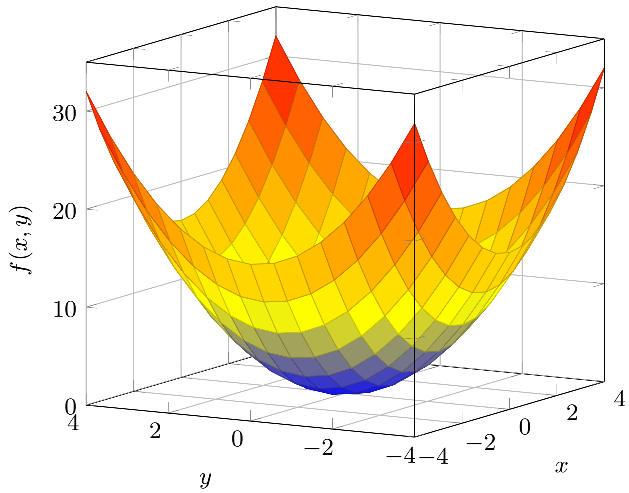
APPENDIX B
LINEARIZATION OF A CONVEX QUADRATIC CONSTRAINT

The quadratic constraints for branch ampacities are in the following form:

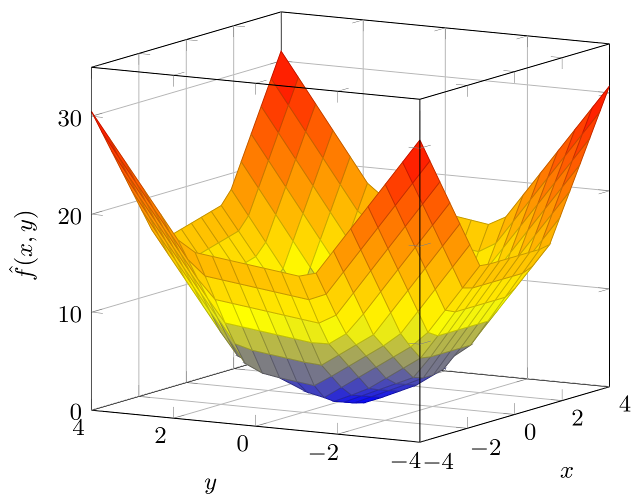
$$x^2 + y^2 \leq r^2 \tag{23}$$

which is the area inside a circle, centered at the origin, with radius r. In order to linearize this constraint, a regular convex polygon approximation of the circle is used here. A regular convex polygon is a polygon where each side has the same length, and all interior angles are equal and less than 180 degrees. Figure 4 shows an inner hexagon approximation of a circle with radius r_x. Define the size of this polygon with the radius r_x and note that with a known r_x and the number of sides, a polygon can be uniquely defined. Take the original circle to be approximated with $r = r_{\text{in}}$. If one takes $r_x = r_{\text{in}}$, the whole hexagon falls inside the circle, which leads to conservative results since some of the possible solutions are omitted. Taking $r_x = r_{\text{out}}$, the whole circle (r_{in}) falls inside the hexagon. In this case, some non-solutions are taken as solutions. Assume that the outer circle (r_{out}) represents the absolute maximum ampacity, while the inner circle (r_{in}) is the maximum ampacity considering a safety margin, e.g., 5%. This allows for a small portion of the solution to exceed the conservative margin without jeopardizing the feeder ampacity. A compromised approximation would be a polygon that falls somewhere in between the two circles of r_{out} and r_{in}. This polygon is shown in Fig. 4 with r_x. The optimal polygon in this case would be the one that makes the two areas in Fig. 4, i.e. A_1 and A_2, equal. The area A_1 shows the omitted solutions and the area A_2 shows the solutions exceed the safety margin. Mathematically, for a generic polygon with n sides, these areas are calculated as follows:

$$A_1 = r_{\text{in}}^2 \left[\cos^{-1}(R) - R\sqrt{1-R^2}\right], \; R = \frac{r_x}{r_{\text{in}}}\cos(\frac{\pi}{n}) \tag{24}$$



(a) Quadratic function



(b) PWL approximation

Figure 3. A standard quadratic function of two variables and its PWL approximation.

$$A_2 = A_1 + \frac{1}{2} r_x^2 \sin\left(\frac{2\pi}{n}\right) - r_{\text{in}}^2 \frac{\pi}{n} \tag{25}$$

In order to find the radius of the optimal polygon, A_1 should be equal to A_2. With this assumption, it follows immediately from (25) that

$$r_x = r_{\text{in}} \sqrt{\frac{\frac{2\pi}{n}}{\sin(\frac{2\pi}{n})}} \tag{26}$$

The error generated by this approximation for different sizes of polygons is calculated by

$$\eta = \frac{n\, A_1}{\pi\, r_{\text{in}}^2} \tag{27}$$

This error is shown in Fig. 5 versus the size of polygon. Using a higher-order polygon leads to lower errors in the approximation. It should be noted, however, that increasing the order of approximation translates into a greater number of constraints in the optimization problem. Therefore, one should

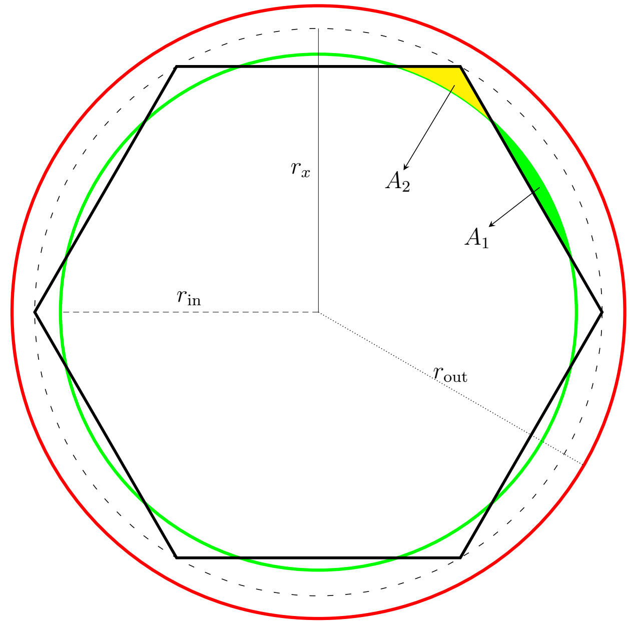


Figure 4. Hexagon approximations of a circle.

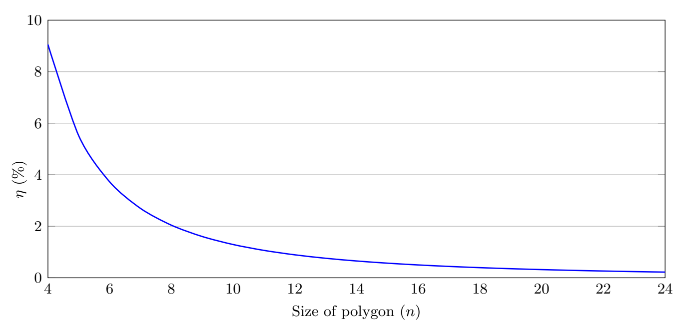


Figure 5. Error of polygon approximation of a circle.

compromise between the approximation error and size of the optimization problem. Based on the authors' experience, using hexagon approximation (i.e. $n = 6$) makes a good balance between the approximation error and the size of the problem. When an hexagon is used in the approximation, the following constraints replace the quadratic constraint in (23):

$$-\sqrt{3}(x + r) \leq y \leq -\sqrt{3}(x - r) \tag{28a}$$

$$-\frac{\sqrt{3}}{2} r \leq y \leq \frac{\sqrt{3}}{2} r \tag{28b}$$

$$\sqrt{3}(x - r) \leq y \leq \sqrt{3}(x + r) \tag{28c}$$

where the radius r can be optimally chosen using (26).

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