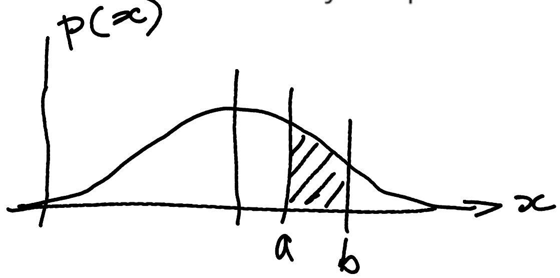
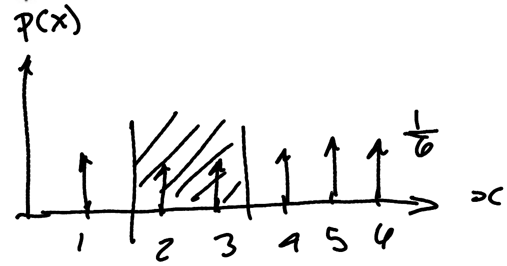
# Information and Capacity

**Exercise 1**: How would you represent a discrete r.v. in a pdf?

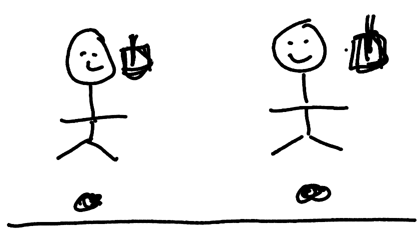



**Exercise 2**: Is the radio noise generated by the sun a stationary stochastic process? Under what conditions?

yes, over short periods of time.

**Exercise 3**: Would the amount of data transmitted by cellular subscribers be an ergodic stochastic process?



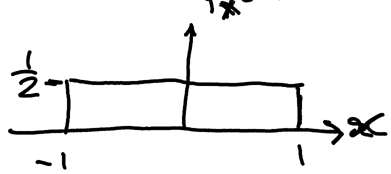
No - no reason to expect same statistics in time as across users.

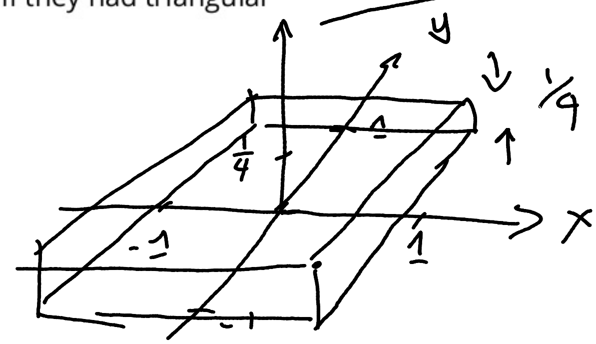
**Exercise 4**: Describe the shape of the joint pdf of two zero-mean iid random variables with uniform pdfs. What if they had triangular pdfs extending between $\pm 1$?

$P_{XY}(x,y) = P_X(x)\,P_Y(y)$

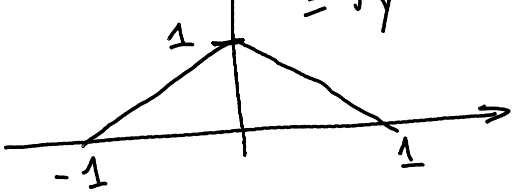
independent

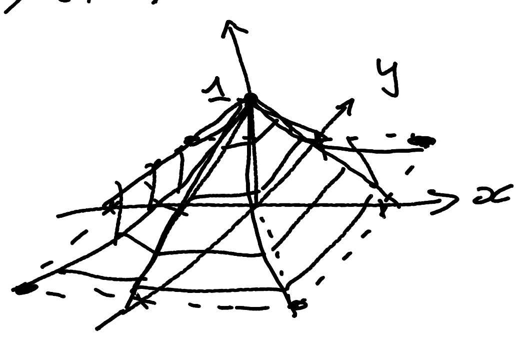
$P_X(x) = P_Y(y)$




$P_X(x) = P_Y(x)$

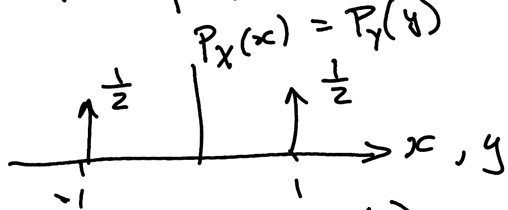


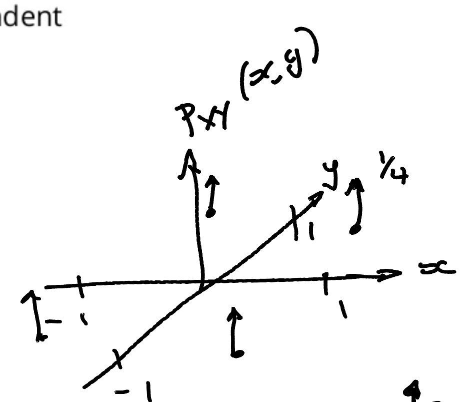
**Exercise 5**: Two random variables, $X$ and $Y$ represent two flips of a coin (outcomes are H or T for each). Draw the joint pdf if the two coins are fair (unbiased) and the outcomes are independent. Draw the joint pdf if the H is twice as likely as T but the outcomes are independent. Draw the joint pdf if the coins are fair but the outcome of the second toss depends on the first and is always the opposite. Which of these are identically distributed? Which are independent r.v.? Which are i.i.d.?

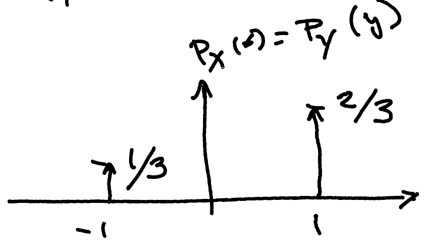
first toss $x$
second toss $y$

map $H \to +1$
$T \to -1$

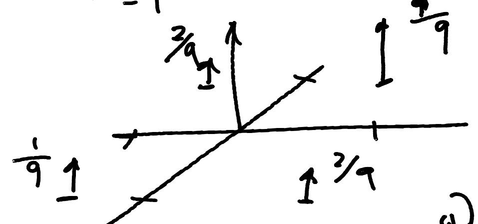


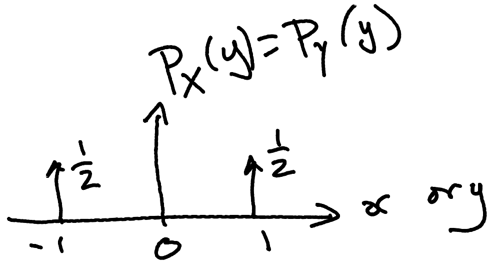

iid. $\longrightarrow$



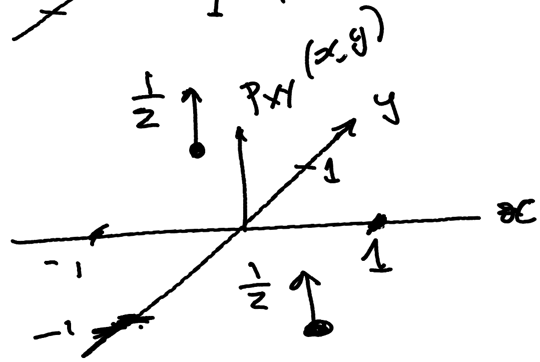




iid. $\longrightarrow$

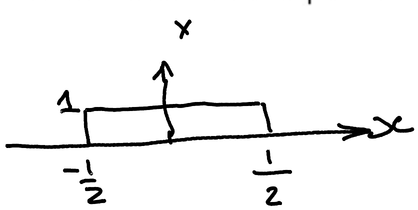


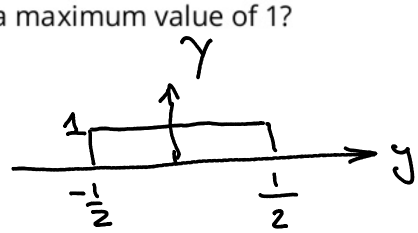



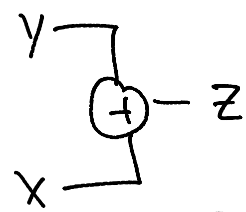

identically distributed but not indpt



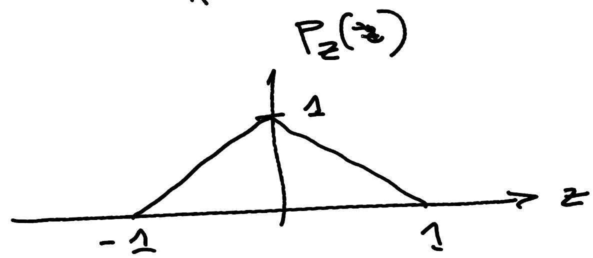

**Exercise 6**: What is the pdf of the sum of two zero-mean iid uniformly-distributed rv's whose pdf has a maximum value of 1?




$P_Z(z) =$



$\bar{X} = \bar{N} = 0$

**Exercise 7**: Prove this.

$$E[(X+N)^2] = E[X^2 + 2XN + N^2]$$

$$= E[X^2] + 2\underbrace{E[XN]}_{=0} + E[N^2]$$

because $\bar{X} = 0$

because uncorrelated

$$= \sigma_X^2 + \sigma_N^2.$$

**Exercise 8**: We observe a source that outputs letters. Out of 10,000 letters 1200 were 'E'. What would be a reasonable estimate of the probability of the letter 'E'?

$$\frac{1200}{10{,}000} \approx 12\%$$

**Exercise 9**: A source generates four different messages. The first three have probabilities 0.125, 0.125, 0.25. What is the probability of the fourth message? How much information is transmitted by each message? What is the entropy of the source? What is the average information rate if 100 messages are generated every second? What if there were four equally-likely messages?

$$\frac{1+1+2}{8} = \frac{4}{8}$$

$$P_i = \frac{1}{8}, \frac{1}{8}, \frac{1}{4}, \frac{1}{2} \qquad \sum_i P_i = 1 \qquad P_3 = 1 - \frac{4}{8} = \frac{1}{2}$$

$$P_3 = \frac{1}{2}$$

$$\frac{1}{8} = 2^{-3} \qquad \log_2\left(\frac{1}{8}\right) = -3$$

$$\frac{1}{4} = 2^{-2} \qquad -2$$

$$\frac{1}{2} = 2^{-1} \qquad -1$$

$$I_i = -\log_2 P_i$$

$$H = \sum_{i=0}^{3} (-\log_2 P_i) P_i$$

$$H = 3 \cdot \frac{1}{8} + 3 \cdot \frac{1}{8} + 2 \cdot \frac{1}{4} + 1 \cdot \frac{1}{2}$$

$$= \frac{3+3+4+4}{8} = \frac{14}{8} = 1.75 \ \underline{\text{bits/message}}$$

$$1.75 \cdot \frac{\text{bits}}{\text{messages}} \times 100 \ \frac{\text{messages}}{\text{s}} = 175 \text{ bps}$$

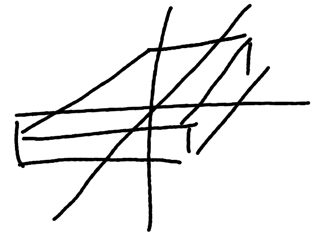
$\leftarrow$

$$P_i = \frac{1}{4}$$

$$I_i = 2 \text{ bits/message}$$

$$H = 2 \quad \text{"}$$

information rate = 200 bps

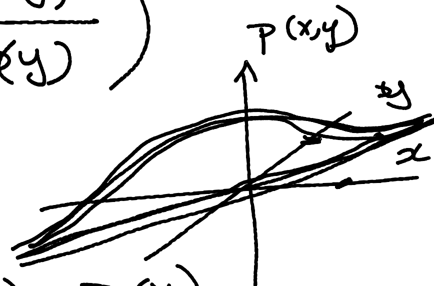


**Exercise 10**: What is the mutual information if $X$ and $Y$ are independent? If they are the same?

$$\mathrm{I}(X;Y) = \sum_{y\in\mathcal{Y}} \sum_{x\in\mathcal{X}} p(x,y) \log_2\left(\frac{p(x,y)}{p(x)\,p(y)}\right)$$

(1) independent: $p(x,y) = p(x) \cdot p(y)$

$$I(X;Y) = \sum \sum p(x)p(y) \log_2 \underbrace{\left(\frac{p(x)\cdot p(y)}{p(x)\cdot p(y)}\right)}_{=1}$$



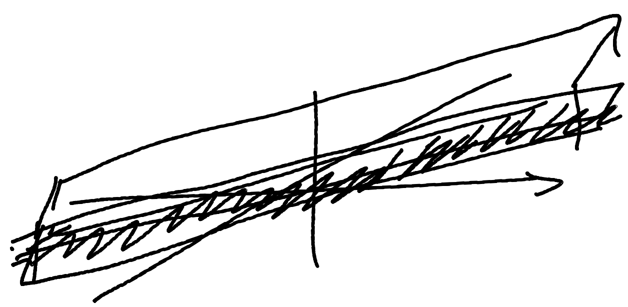

$= 0$

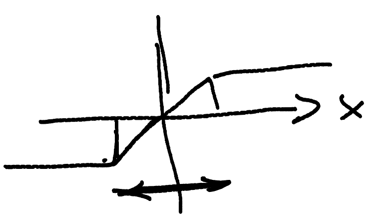
(2) same: $X = y$ $\quad p(x,y) = p(x) = p(y)$

$$\sum \sum p(x) \log_2\left(\frac{\cancel{p(x)}}{p^2(x)}\right)$$

$$= \sum \sum p(x)\left(-\log_2(p(x))\right)$$

$= H(X)$





**Exercise 11**: What is capacity of a binary channel with a BER of $\frac{1}{8}$ (assuming the same BER for 0's and 1's)?

$$\log_2 \frac{1}{8} = -3$$

$$\log_2 \frac{7}{8} = -0.2$$

$$C = 1 - (-p \log_2 p - (1-p)\log_2(1-p))$$

$$= 1 - \left(-\frac{1}{8}\cdot(-3) - \left(\frac{7}{8}\right)\log_2\left(\frac{7}{8}\right)\right)$$

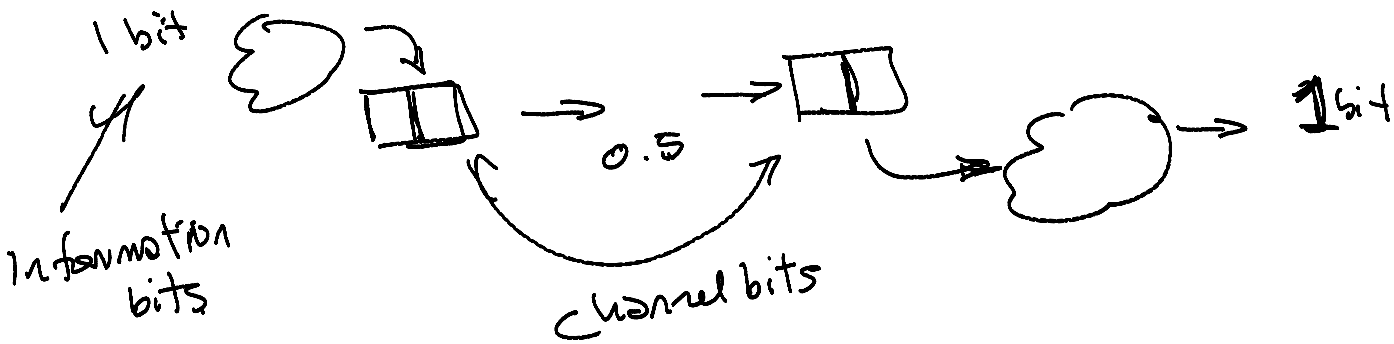
$$= 1 - \left(\frac{3}{8} + \frac{7}{8}\cdot 0.2\right)$$

$$= 1 - \left(\frac{3 + 1.4}{8}\right) \approx 0.5$$

information
channel

if transmit < 0.5 bits/bit

we can make error rate → 0.




(bits/ channel use.)

**Exercise 12**: What is the channel capacity of a 4 kHz channel with an SNR of 30dB?

$B = 4\,\text{kHz}$

$$\frac{S}{N} = 30\,\text{dB} = 10^{\frac{30}{10}} = 1000$$

$$C = B \log_2\left(1 + \frac{S}{N}\right)$$

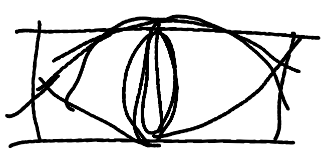
$$= 4000 \cdot \underbrace{\log_2(1 + 1000)}_{\approx 10}$$

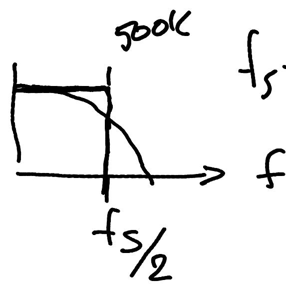
$$= 40\ \text{kbit/s}.$$

**Exercise 13**: Can we achieve capacity by transmitting an NRZ waveform?

No, NRZ has a discrete ($\pm 1$) distribution (not Gaussian).

**Exercise 14**: What do the Nyquist no-ISI criteria and the Shannon Capacity Theorem limit? What channel parameters determine these limits?






$f_s = 1\,\text{MHz}$

Nyquist no-ISI symbol rate bandwidth

Shannon Capacity error-free information rate. joint & marginal probabilities.

**Exercise 15**: You receive 1 million frames, each of which contains 100 bits. By comparing the received frames to the transmitted ones you find that 56 frames had errors. Of these, 40 frames had one bit in error, 15 had two bit errors and one had three errors. What was the FER? The BER?

$10^6$ frames of 100 bits

56 had 1+ errors

40 had 1 error

15 had 2 error

1 had 3 errors.

$$FER = ? \quad \frac{56}{10^6} = 56 \times 10^{-6}$$

$$BER = ? \quad \frac{1 \times 40 + 2 \times 15 + 1 \times 3}{100 \times 10^6} = \frac{40 + 30 + 3}{100 \times 10^6} =$$

$$7.3 \times 10^{-7}$$