Information and Capacity

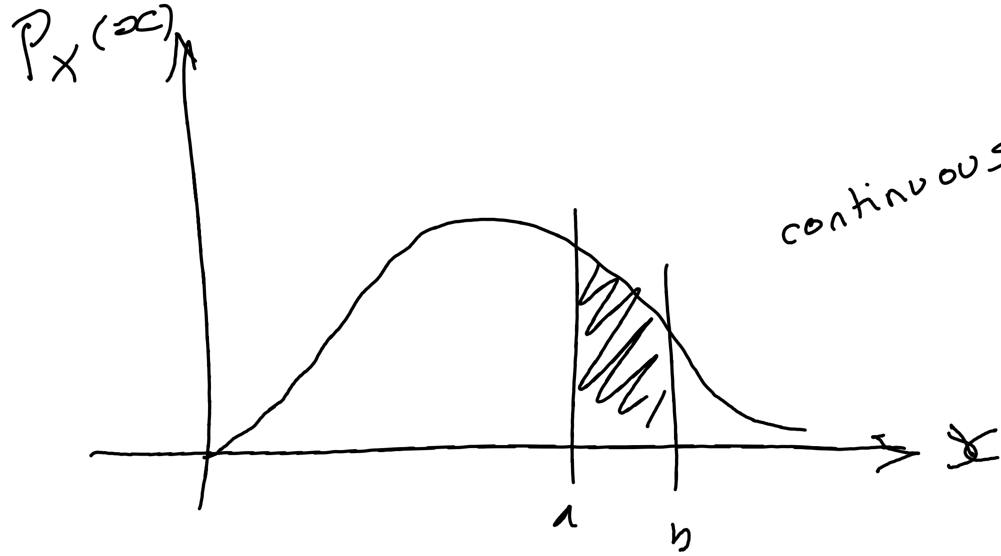
Exercise 1: Give an example of a communication system. If you can, identify the source, transmitter, channel, receiver and destination.

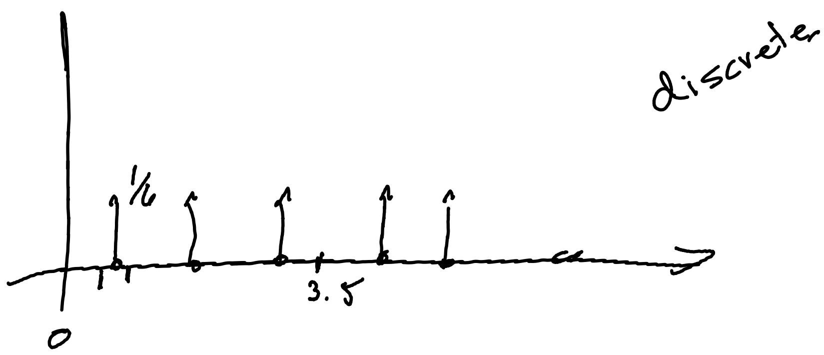
smoke signals

person → fire/blanket → free space → eyes → brain

↑ sunlight

Exercise 2: How would you represent a discrete r.v. in a pdf?





Exercise 3: Is the radio noise generated by the sun a stationary stochastic process? Under what conditions?

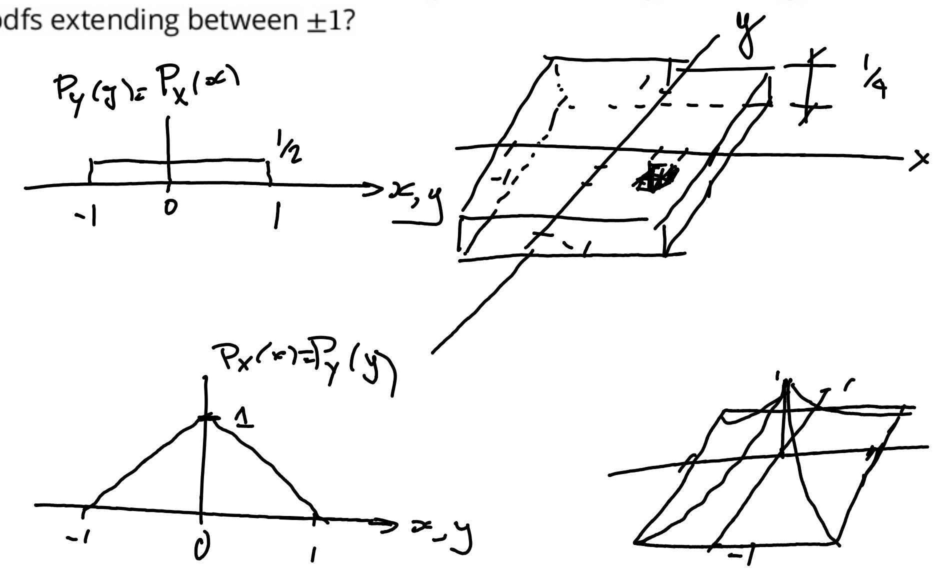
yes – short periods

no – day vs. night (e.g.)

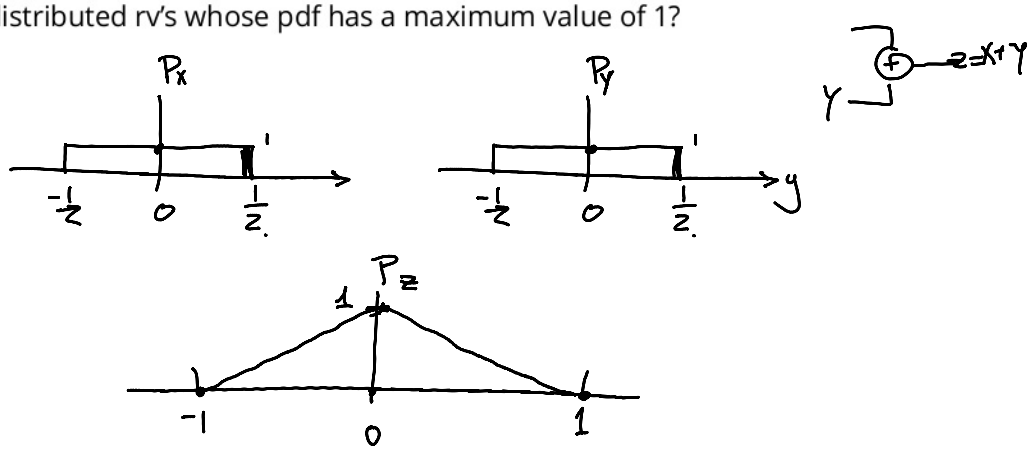
Exercise 4: Would the amount of data transmitted by cellular subscribers be an ergodic stochastic process?

No, probably not.

Exercise 5: Describe the shape of the joint pdf of two zero-mean iid random variables with uniform pdfs. What if they had triangular pdfs extending between ± 1?



Exercise 6: What is the pdf of the sum of two zero-mean iid uniformly-distributed rv's whose pdf has a maximum value of 1?



Exercise 7: Prove this.

$$E\left[(X+Y)^2\right] = E\left[X^2 + 2XY + Y^2\right]$$

$$\underline{E[X^2]} + E[2XY] + \underline{E[Y^2]}$$

$$\downarrow$$

$$2\,\cancel{E[XY]} \rightarrow 0 \quad \text{(uncorrelated)}$$

Exercise 8: We observe a source that outputs letters. Out of 10,000 letters 1200 were 'E'. What would be a reasonable estimate of the probability of the letter 'E'?

estimate of $\quad P_L(l = \text{'E'}) = \dfrac{1200}{10{,}000} = 12\%$

Exercise 9: A source generates four different messages. The first three have probabilities 0.125, 0.125, 0.25. What is the probability of the fourth message? How much information is transmitted by each message? What is the entropy of the source? What is the average information rate if 100 messages are generated every second? What if there were four equally-likely messages?

$\frac{1}{8}, \frac{1}{8}, \frac{1}{4}$ $\quad P_{M_3} = 1 - \left(\frac{1}{8} + \frac{1}{8} + \frac{1}{4}\right) = \frac{1}{2}$

$I_0 = I_1 = -\log_2\left(\frac{1}{8}\right) = 3$ bits

$I_2 = -\log_2\left(\frac{1}{4}\right) = 2$ bits

$I_3 = -\log_2\left(\frac{1}{2}\right) = 1$ bit

$H = \frac{1}{8} \cdot 3 + \frac{1}{8} \cdot 3 + \frac{1}{4} \cdot 2 + \frac{1}{2} \cdot 1 = \frac{6}{8} + \frac{4}{8} + \frac{4}{8}$

$= \frac{14}{8} = 1.75$ bits/message

for 100 messages/second

$H \cdot R$ (rate) $= 100 \cdot 1.75 = 175$ bits/second

for $P_i = \frac{1}{4}$ $\quad H = 2$ bits/msg

$H \cdot R = 200$ bits/s

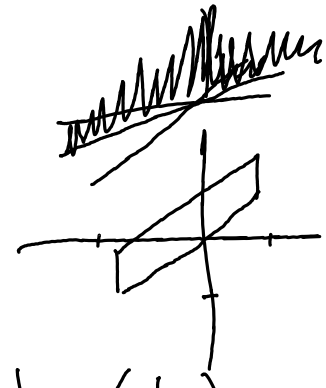
Exercise 10: What is the mutual information if X and Y are independent? If they are the same?

independent $\quad p(x,y) = p(x)\,p(y)$

$$I(x) = \sum \sum p(x,y) \cancel{\log_2\left(\frac{p(x,y)}{p(x)\,p(y)}\right)}$$

$\log_2(1) = 0$

$$= \sum \sum p(x,y)\; 0 = 0$$



same $\quad p(x,y) = p(x) = p(y)$

$$= \sum \sum \underbrace{p(x,y)}_{p(x)} \log_2\left(\frac{p(x)}{p(x)\,p(y)}\right)$$

$$= H(X)$$

$$\log_2\left(\frac{1}{p(x)}\right)$$

$$= -\log_2(p(x))$$

$$= I(p(x))$$

Exercise 11: What is capacity of a binary channel with a BER of $\frac{1}{8}$ (assuming the same BER for 0's and 1's)?

$\log_2 \frac{1}{8} = -3$

$\log_2 \frac{7}{8} = -0.2$

$$C = 1 - (-p \log_2 p - (1-p)\log_2(1-p))$$

$p = \frac{1}{8}$

$$C = 1 - \left(-\frac{1}{8}(-3) - \left(\frac{7}{8}\right)(-0.2)\right)$$

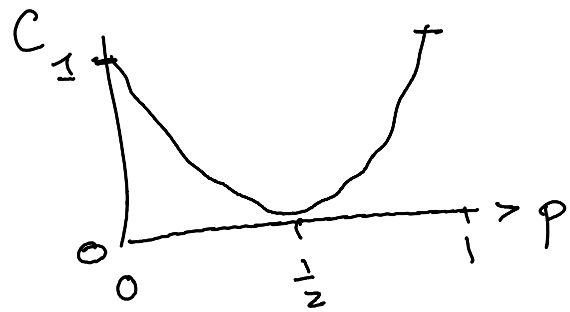
$$= 1 - \left(\frac{3}{8} + \frac{1.4}{8}\right) \simeq 1 - \frac{4.8}{8} \approx 0.38$$

bits/channel use.

$p = \frac{1}{2}$ $\quad C = 0$

$p = 1$ $\quad C = 1$

$p = 0$ $\quad C = 1$



Exercise 12: What is the channel capacity of a 4 kHz channel with an SNR of 30dB?

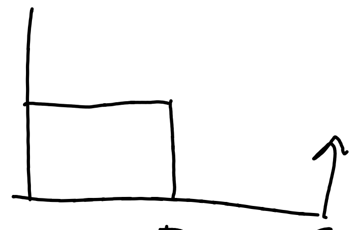
$10^{\frac{30}{10}} = 1000$

$$C = B \log_2\left(1 + \frac{S}{N}\right)$$

$$= 4 \times 10^3 \log_2(1 + 1000)$$

$$\approx 40 \text{ kb/s}$$

Exercise 13: What do the Nyquist no-ISI criteria and the Shannon Capacity Theorem limit? What channel parameters determine these limits?



B $\quad f_s =$ maximum signal rate without ISI

- Nyquist: limits the number of symbols/second without ISI

is a function of bandwidth

- Shannon Capacity: number of information bits per channel use

is a function of mutual information

e.g.: if $B = 4$ kHz

$f_{no\text{-}ISI} = 8$ kHz

Exercise 14: You receive 1 million frames, each of which contains 100 bits. By comparing the received frames to the transmitted ones you find that 56 frames had errors. Of these, 40 frames had one bit in error, 15 had two bit errors and one had three errors. What was the FER? The BER?

10^6 frames $\cdot$ 100 bits/frame $= 10^8$ bits

$$56 \left\{ \begin{matrix} 40 \times 1 \\ 15 \times 2 \\ 1 \times 3 \end{matrix} \right. \rightarrow \begin{matrix} \text{FER} = 56 \times 10^{-6} \\ \text{BER} = 73 \times 10^{-8} \end{matrix}$$

bit errors $= 40 + 30 + 3 = 73$

bits $= 10^8$