Measurement-Based Estimation of the Power Flow Jacobian Matrix

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***Abstract*—In this paper, we propose a measurement-based method to compute the power flow Jacobian matrix, from which we can infer pertinent information about the system topology in near real-time. A salient feature of our approach is that it readily adapts to changes in system operating point and topology; this is desirable as it provides power system operators with a way to update, as the system evolves, the models used in many reliability analysis tools. The method uses high-speed synchronized voltage and current phasor data collected from phasor measurement units to estimate entries of the Jacobian matrix through linear total least-squares (TLS) estimation. In addition to centralized TLS-based algorithms, we provide distributed alternatives aimed at reducing computational burden. Through numerical case studies, we illustrate the effectiveness of our proposed Jacobian-matrix estimation approach as compared to the conventional model-based one.**

***Index Terms*—Power flow Jacobian matrix, real-time monitoring, phasor measurement units, sensitivity.**

I. Introduction

Power flow analysis is an important tool for power system planning and operations. For example, it enables operators to assess whether or not, under quasi steady-state conditions, a power system satisfies certain basic operational requirements, e.g., bus voltage magnitudes remain close to rated values, and transmission lines are not overloaded [1]. The power flow problem is often solved via the iterative Newton-Raphson algorithm, which simultaneously solves a set of nonlinear equations with an equal number of unknowns [1]. At each iteration, the algorithm considers a linearized problem constructed from the power flow Jacobian matrix, which is a sparse matrix that results from a sensitivity analysis of the power flow equations. Also, in real-time contingency analysis (RTCA), by solving the power flow equations repeatedly for all credible contingency scenarios, operators determine whether or not the system will meet operational reliability requirements in case of outage in any one particular asset (e.g., a generator or a transmission line), a condition known as N-1 security. In addition to its direct use in the numerical solution to the power flow problem, the eigenvalues of the Jacobian matrix have long been used as indices of system vulnerability to voltage instabilities [2]. Also, since the sparsity structure of the Jacobian matrix is closely related to the graph Laplacian of the underlying network, the structure of the estimated Jacobian matrix inherently contains the most up-to-date network topology and corresponding parameters; thus, it is useful in other online analysis tools, such as state estimation.

In order to monitor and maintain operational reliability using the analysis tools described above, power system operators must rely on an accurate power flow Jacobian matrix, which may be out-of-date due to erroneous records, faulty telemetry from remotely monitored circuit breakers, or unexpected operating conditions resulting from, e.g., unforeseen equipment failure. Therefore, in this paper, we propose a method for online estimation of the power flow Jacobian matrix using only measurements obtained from phasor measurement units (PMUs), which measure voltages, currents, and frequency at a very high speed (usually 30 measurements per second) [3].

Our approach to online Jacobian matrix estimation builds upon our previous work in [4], [5]. In [4], by relying on active power bus injection and line flow data obtained from PMUs, linear sensitivity distribution factors are computed via the solution of a linear least-squares errors (LSE) estimation problem. In this paper, by exploiting slight fluctuations in measurements of bus voltage magnitudes and phase angles, as well as those of net active and reactive power injections obtained from PMUs, we construct an overdetermined set of linear equations, and solve it via total least-squares (TLS) estimation; the solution to the problem provides the entries of the Jacobian matrix. In this regard, in [4], even though the regressor matrix is constructed from PMU measurements, it is assumed to be error-free as per the LSE estimation framework. In contrast, the TLS-based estimation method proposed in this paper to compute the Jacobian matrix accounts for errors present in both the regressor matrix and the observation vector. Furthermore, we improve the adaptability of the proposed method by formulating a weighted TLS (WTLS) problem, in which recent measurements are weighted more favorably than past ones. We illustrate the effectiveness of the proposed measurement-based Jacobian estimation method by comparing its results to benchmark values obtained via direct linearization of the power flow equations at a particular operating point. The estimated Jacobian matrix is quite accurate and can therefore be used in studies that rely on the power flow model.

Another advantage of our proposed approach is that an up-to-date estimate of the Jacobian matrix can be used to infer the current network topology and pertinent parameter values. Topology errors have long been cited as a cause of inaccurate state estimation results [6]. Since then, numerous approaches

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have been proposed to detect and identify topology errors in the context of state estimation [7]–[9]. The specific issue of determining external system topology errors was explored in [10]. In [11], the state estimation problem is reformulated as a least absolute value optimization problem, in order to determine whether a line exists between two buses. Owing to the potential improvement in situational awareness held by the widespread deployment of PMUs, recent work has focused on external system line-outage detection and/or identification by taking advantage of voltage phase angle measurements [12]–[15]. Recently, in [16], synchrophasor data is used to identify an equivalent power system network, which includes only buses that are equipped with phasor measurement capabilities, while employing the so-called DC power flow approximations. While the methods proposed in the aforementioned works may provide updated network connectivity information, they do not offer further insight with regard to associated parameters, e.g., transmission line conductance and admittance.

For the most part, we assume an offline model entirely unavailable and that all buses within the monitored region are equipped with PMUs. Admittedly, present-day power systems are still far from having such a wide coverage of the network. However, incentives to invest in measurement infrastructure are driven by preliminary demonstrations of its potential benefits in monitoring, protection, and control capabilities [17]. Moreover, today, in addition to standalone PMU installations, synchronous phasor measurement capabilities are available as standard features in many protective relays, meters, and recorders [18]. Thus, we foresee that a sufficiently rich measurement set will be available in the near future.

The remainder of this paper is organized as follows. Section II describes the conventional model-based method to obtain the power flow Jacobian matrix, and formulates the proposed measurement-based method as the solution of a TLS problem. In Section III, we describe algorithms to solve the basic and weighted TLS problems and illustrate them via examples. In Section IV, we refine the problem formulation so as to enable a distributed implementation of the proposed TLS-based Jacobian estimation approach. Case studies involving the IEEE 118-bus test system are presented in Section V. Finally, we provide concluding remarks in Section VI.

II. Preliminaries

Let $\mathcal{N}$ denote the set of N buses in the system. Let V_i and θ_i, respectively, denote the voltage magnitude and phase angle at bus i; additionally, let P_i and Q_i, respectively, denote the net active and reactive power injections at bus i. The entries of the power flow Jacobian matrix are composed of partial derivatives of P_i with respect to θ_j and V_j, which we denote by Ψ_i^j and Φ_i^j, respectively, and partial derivatives of Q_i with respect to θ_j and V_j, which we denote by Γ_i^j and Λ_i^j, respectively. Suppose θ_j varies by a small amount, denoted by $\Delta\theta_j$. Also denote by $\Delta P_i^{\theta_j}$ the change in active power injection at bus i, resulting from $\Delta\theta_j$, with all other system quantities held constant. Then, it follows that

$$\Psi_i^j := \frac{\partial P_i}{\partial \theta_j} \approx \frac{\Delta P_i^{\theta_j}}{\Delta \theta_j}. \tag{1}$$

On the other hand, suppose V_j varies by a small amount, denoted by ΔV_j. Also denote by $\Delta P_i^{V_j}$ the change in active power injection at bus i, resulting from ΔV_j, with all other system quantities held constant. Then, it follows that

$$\Phi_i^j := \frac{\partial P_i}{\partial V_j} \approx \frac{\Delta P_i^{V_j}}{\Delta V_j}. \tag{2}$$

Similarly, we define the analogue of (1)–(2) for reactive power as follows:

$$\Gamma_i^j := \frac{\partial Q_i}{\partial \theta_j} \approx \frac{\Delta Q_i^{\theta_j}}{\Delta \theta_j}, \tag{3}$$

where $\Delta Q_i^{\theta_j}$ denotes the change in reactive power injection at bus i, resulting from $\Delta\theta_j$, with all other quantities held constant; and

$$\Lambda_i^j := \frac{\partial Q_i}{\partial V_j} \approx \frac{\Delta Q_i^{V_j}}{\Delta V_j}, \tag{4}$$

where $\Delta Q_i^{V_j}$ denotes the change in reactive power injection at bus i, resulting from ΔV_j. Traditionally, the sensitivity factors in (1)–(4) have been computed offline based on a model of the power system, including its topology and pertinent parameters. Next, we describe this traditional model-based approach.

A. Model-Based Approach to Jacobian Computation

Consider a power system with N buses, each of which is categorized into one of the following: (i) slack bus, for which the voltage magnitude is fixed and with respect to which the phase angles of all other buses are measured, (ii) voltage-controlled (PV) bus, for which the voltage magnitude is fixed, or (iii) load (PQ) bus, for which neither voltage magnitude nor phase angle are fixed (see, e.g., [1]). Let $\mathcal{N}_L$ ($\mathcal{N}_G$) denote the set of N_L load (N_G voltage-controlled) buses. Furthermore, without loss of generality, in the remainder of this paper, we assume that bus 1 is designated as the slack bus. Then, the static behavior of the power system can be described by the power flow equations:

$$P_i = p_i(\theta_1, \dots, \theta_N, V_1, \dots, V_N), \; i \in \mathcal{N}_G \cup \mathcal{N}_L, \tag{5}$$

and

$$Q_i = q_i(\theta_1, \dots, \theta_N, V_1, \dots, V_N), \; i \in \mathcal{N}_L. \tag{6}$$

In (5)–(6), the dependence on network parameters, such as line and shunt impedances, is implicitly considered in $p_i(\cdot)$ and $q_i(\cdot)$. Suppose a solution to (5)–(6) exists at $(\theta_i^0, V_i^0, P_i^0, Q_i^0)$, $i = 1, \dots, N$. Further, assume $p_i(\cdot)$, for all $i \in \mathcal{N}_G \cup \mathcal{N}_L$, and $q_i(\cdot)$, for all $i \in \mathcal{N}_L$, are continuously differentiable with respect to θ_i and V_i, for all $i = 1, \dots, N$, at $(\theta_i^0, V_i^0, P_i^0, Q_i^0)$, $i = 1, \dots, N$. For each i, let $\theta_i = \theta_i^0 + \Delta\theta_i$, $V_i = V_i^0 + \Delta V_i$, $P_i = P_i^0 + \Delta P_i$, and $Q_i = Q_i^0 + \Delta Q_i$. Then, assuming $\Delta\theta_i$, ΔV_i, ΔP_i, and ΔQ_i are sufficiently small, we can approximate (5) as

$$\begin{aligned} P_i^0 + \Delta P_i \approx{}& p_i(\theta_1^0, \dots, \theta_N^0, V_1^0, \dots, V_N^0) \\ &+ \sum_{j \in \mathcal{N}_G \cup \mathcal{N}_L} \Psi_i^j \Delta\theta_j + \sum_{j \in \mathcal{N}_L} \Phi_i^j \Delta V_j, \end{aligned} \tag{7}$$

TABLE I: WECC 3-machine 9-bus systems—model- and measurement-based sensitivity factors obtained in Examples 1 and 2.

Ψ_5^2	Ψ_5^3	Ψ_5^4	Ψ_5^5	Ψ_5^6	Ψ_5^7	Ψ_5^8	$\hat{\Psi}_5^2$	$\hat{\Psi}_5^3$	$\hat{\Psi}_5^4$	$\hat{\Psi}_5^5$	$\hat{\Psi}_5^6$	$\hat{\Psi}_5^7$	$\hat{\Psi}_5^8$
0	0	-10.86	0	16.54	0	0	0.07631	0.05691	-11.03	0.03587	16.64	-0.06743	-0.02094
Ψ_5^9	Φ_5^4	Φ_5^5	Φ_5^6	Φ_5^7	Φ_5^8	Φ_5^9	$\hat{\Psi}_5^9$	$\hat{\Phi}_5^4$	$\hat{\Phi}_5^5$	$\hat{\Phi}_5^6$	$\hat{\Phi}_5^7$	$\hat{\Phi}_5^8$	$\hat{\Phi}_5^9$
-5.6816	-2.239	0	2.3762	0	0	-1.8495	-5.7703	-2.236	-0.02054	2.557	-0.08047	0.03681	-2.017
Γ_5^2	Γ_5^3	Γ_5^4	Γ_5^5	Γ_5^6	Γ_5^7	Γ_5^8	$\hat{\Gamma}_5^2$	$\hat{\Gamma}_5^3$	$\hat{\Gamma}_5^4$	$\hat{\Gamma}_5^5$	$\hat{\Gamma}_5^6$	$\hat{\Gamma}_5^7$	$\hat{\Gamma}_5^8$
0	0	0	0	0	2.4279	-3.861	-0.05818	0.02315	-0.1354	0.02593	0.06445	2.519	-3.855
Γ_5^9	Λ_5^4	Λ_5^5	Λ_5^6	Λ_5^7	Λ_5^8	Λ_5^9	$\hat{\Gamma}_5^9$	$\hat{\Lambda}_5^4$	$\hat{\Lambda}_5^5$	$\hat{\Lambda}_5^6$	$\hat{\Lambda}_5^7$	$\hat{\Lambda}_5^8$	$\hat{\Lambda}_5^9$
1.433	0	0	0	-13.81	23.33	-9.912	1.363	-0.04853	-0.006372	0.02142	-13.75	23.33	-9.890

for each $i \in \mathcal{N}_G \cup \mathcal{N}_L$, and (6) as

$$Q_i^0 + \Delta Q_i \approx q_i(\theta_1^0, \dots, \theta_N^0, V_1^0, \dots, V_N^0) + \sum_{j \in \mathcal{N}_G \cup \mathcal{N}_L} \Gamma_i^j \Delta\theta_j + \sum_{j \in \mathcal{N}_L} \Lambda_i^j \Delta V_j, \quad (8)$$

for each $i \in \mathcal{N}_L$, where

$$\Psi_i^j = \frac{\partial p_i}{\partial \theta_j},\ \Phi_i^j = \frac{\partial p_i}{\partial V_j},\ \Gamma_i^j = \frac{\partial q_i}{\partial \theta_j},\ \text{and}\ \Lambda_i^j = \frac{\partial q_i}{\partial V_j},$$

all of which are evaluated at the nominal operating point $(\theta_i^0, V_i^0, P_i^0, Q_i^0)$, $i = 1, \dots, N$. Note that in (7)–(8), we have accounted for the fact that the voltages at the slack bus and the voltage-controlled buses are fixed. Next, we illustrate the ideas presented above with an example.

Example 1 (3-Machine 9-Bus System): In this example, we consider the WECC 3-machine, 9-bus system model (see, e.g., [19]). In this system, bus 1 is designated as the slack bus; there are $N_G = 2$ voltage-controlled buses, consisting of $\mathcal{N}_G = \{2, 3\}$; and there are $N_L = 6$ load buses, consisting of $\mathcal{N}_L = \{4, 5, \dots, 9\}$. In this example, we compute model-based sensitivity factors by linearizing the power flow equations in (7)–(8). In the left-half portion of Table I, we report the sensitivities of active and reactive power injections at bus 4 with respect to voltage magnitudes and phase angles at all other buses. ■

The traditional model-based approach described above is not ideal since accurate and up-to-date network topology, parameters, and operating point are required. In this paper, we aim to eradicate the reliance on system models in the computation of the sensitivities defined in (1)–(4), and improve adaptability to changes occurring in the system. With regard to this, we propose a method to estimate these sensitivities using only PMU measurements obtained in near real-time without relying on the full nonlinear power flow model of the system.

B. Measurement-Based Approach to Jacobian Computation

Denote the voltage phase angle at bus j at times t and $t+\Delta t$, $\Delta t > 0$ and small, as $\theta_j(t)$ and $\theta_j(t+\Delta t)$, respectively. Also denote the voltage magnitude at bus j at times t and $t+\Delta t$, as $V_j(t)$ and $V_j(t+\Delta t)$, respectively. Define $\Delta\theta_j(t) = \theta_j(t+\Delta t) - \theta_j(t)$ and $\Delta V_j(t) = V_j(t+\Delta t) - V_j(t)$; then, according to the approximations of Ψ_i^j, Φ_i^j, Γ_i^j, and Λ_i^j in (1)–(4), we have that, at time t,

$$\Psi_i^j \approx \frac{\Delta P_i^{\theta_j}(t)}{\Delta\theta_j(t)},\ \Phi_i^j \approx \frac{\Delta P_i^{V_j}(t)}{\Delta V_j(t)}, \quad (9)$$

$$\Gamma_i^j \approx \frac{\Delta Q_i^{\theta_j}(t)}{\Delta\theta_j(t)},\ \text{and}\ \Lambda_i^j \approx \frac{\Delta Q_i^{V_j}(t)}{\Delta V_j(t)}. \quad (10)$$

We assume $\theta_j(t)$, $V_j(t)$, $\theta_j(t+\Delta t)$, and $V_j(t+\Delta t)$ are measurements available from PMUs. As evidenced in (9), in order to compute Ψ_i^j and Φ_i^j, we also need $\Delta P_i^{\theta_j}(t)$ and $\Delta P_i^{V_j}(t)$, which are not readily available from PMU measurements. However, we assume that the total variation in net active power injection at bus i (i.e., $\Delta P_i(t)$) is available from PMU measurements. We express this total variation as the sum of active power injection variations at bus $i \in \mathcal{N}_G \cup \mathcal{N}_L$ due to variations in voltage phase angle $j \in \mathcal{N}_G \cup \mathcal{N}_L$ and magnitude at each bus $j \in \mathcal{N}_L$:

$$\Delta P_i(t) \approx \sum_{j \in \mathcal{N}_G \cup \mathcal{N}_L} \Delta P_i^{\theta_j}(t) + \sum_{j \in \mathcal{N}_L} \Delta P_i^{V_j}(t). \quad (11)$$

Similarly, from (10), we note that in order to compute Γ_i^j and Λ_i^j, we need $\Delta Q_i^{\theta_j}(t)$ and $\Delta Q_i^{V_j}(t)$, which are not readily available from PMU measurements. By making similar assumptions to the ones used in the derivation of (11), we express the net variation in net reactive power injection at bus i as

$$\Delta Q_i(t) \approx \sum_{j \in \mathcal{N}_G \cup \mathcal{N}_L} \Delta Q_i^{\theta_j}(t) + \sum_{j \in \mathcal{N}_L} \Delta Q_i^{V_j}(t). \quad (12)$$

Now, by substituting (9) into (11), we can express (11) as

$$\Delta P_i(t) \approx \sum_{j \in \mathcal{N}_G \cup \mathcal{N}_L} \Delta\theta_j(t)\Psi_i^j + \sum_{j \in \mathcal{N}_L} \Delta V_j(t)\Phi_i^j,$$

where $\Psi_i^j \approx \frac{\Delta P_i^{\theta_j}}{\Delta\theta_j}$ and $\Phi_i^j \approx \frac{\Delta P_i^{V_j}}{\Delta V_j}$. Analogously, by substituting (10) into (12), we can express (12) as

$$\Delta Q_i(t) \approx \sum_{j \in \mathcal{N}_G \cup \mathcal{N}_L} \Delta\theta_j(t)\Gamma_i^j + \sum_{j \in \mathcal{N}_L} \Delta V_j(t)\Lambda_i^j,$$

where $\Gamma_i^j \approx \frac{\Delta Q_i^{\theta_j}}{\Delta\theta_j}$ and $\Lambda_i^j \approx \frac{\Delta Q_i^{V_j}}{\Delta V_j}$.

Suppose $M+1$ sets of synchronized measurements are available. Let

$$\begin{aligned}
\Delta P_i[k] &= P_i((k+1)\Delta t) - P_i(k\Delta t), \\
\Delta Q_i[k] &= Q_i((k+1)\Delta t) - Q_i(k\Delta t), \\
\Delta\theta_i[k] &= \theta_i((k+1)\Delta t) - \theta_i(k\Delta t), \\
\Delta V_i[k] &= V_i((k+1)\Delta t) - V_i(k\Delta t),
\end{aligned}$$

$k = 1, \dots, M$. Next, define $\Delta P_i = [\Delta P_i[1], \dots, \Delta P_i[M]]^T$ and $\Delta Q_i = [\Delta Q_i[1], \dots, \Delta Q_i[M]]^T$; similarly, define $\Delta\theta_i = [\Delta\theta_i[1], \dots, \Delta\theta_i[M]]^T$ and $\Delta V_i = [\Delta V_i[1], \dots, \Delta V_i[M]]^T$. Then, we obtain the following systems of equations:

$$\Delta P_i \approx \left[(\Delta\theta_j)_{j \in \mathcal{N}_G \cup \mathcal{N}_L} \quad (\Delta V_j)_{j \in \mathcal{N}_L} \right] \begin{bmatrix} \Psi_i \\ \Phi_i \end{bmatrix}, \quad (13)$$

where

$$\Psi_i = \left[\left(\Psi_i^j \right)_{j \in \mathcal{N}_G \cup \mathcal{N}_L} \right] \text{ and } \Phi_i = \left[\left(\Phi_i^j \right)_{j \in \mathcal{N}_L} \right],$$

and

$$\Delta Q_i \approx \left[(\Delta\theta_j)_{j \in \mathcal{N}_G \cup \mathcal{N}_L} \quad (\Delta V_j)_{j \in \mathcal{N}_L} \right] \begin{bmatrix} \Gamma_i \\ \Lambda_i \end{bmatrix}, \tag{14}$$

where

$$\Gamma_i = \left[\left(\Gamma_i^j \right)_{j \in \mathcal{N}_G \cup \mathcal{N}_L} \right] \text{ and } \Lambda_i = \left[\left(\Lambda_i^j \right)_{j \in \mathcal{N}_L} \right].$$

In (13)–(14), we assume that the relationship between ΔP_i and $[\Psi_i^T, \Phi_i^T]^T$ and the one between ΔQ_i and $[\Gamma_i^T, \Lambda_i^T]^T$ are approximately linear. Under this assumption, we seek the best estimate for $[\Psi_i^T, \Phi_i^T]^T$ and $[\Gamma_i^T, \Lambda_i^T]^T$ given the measured observations. It is worth noting that, in some sense, the formulation of (13) and (14) represents the worst-case scenario, in which we assume system topology and parameter information are wholly unavailable. Later, in Section IV, we relax this restriction and expect that, with a priori knowledge of the system, the method proposed in Section III will result in more accurate estimates of the sensitivity factors $[\Psi_i^T, \Phi_i^T]^T$ and $[\Gamma_i^T, \Lambda_i^T]^T$.

C. Problem Statement

Suppose the systems in (13)–(14) are overdetermined, i.e., $M > \bar{N} = N_G + 2N_L$. Then, a natural solution approach is to obtain Ψ_i, Φ_i, Γ_i, and Λ_i via least-squares errors (LSE) estimation. In ordinary LSE estimation, the regressor matrix is assumed to be free of error; hence all errors are confined to the observation vector (in our setting, ΔP_i or ΔQ_i). This assumption, however, is not entirely appropriate in our problem setting, since ΔP_i, ΔQ_i, $\Delta\theta_j$, and ΔV_j are all constructed from PMU measurements obtained in real-time. In such a case where modeling and measurement errors are associated with both the observation vectors and the regressor matrix, total least-squares (TLS) estimation, instead of LSE estimation, is one appropriate method for fitting [20].

III. Total Least-Squares Approach to Jacobian Estimation

In our setting, measurement and modeling errors enter into both the regressor matrix and the observation vectors in (13)–(14). In this section, we formulate the TLS estimation problem and its solution with respect to the system in (13) (the formulation with respect to the system in (14) is analogous). Further, for ease of notation, let

$$A = \left[(\Delta\theta_j)_{j \in \mathcal{N}_G \cup \mathcal{N}_L} \quad (\Delta V_j)_{j \in \mathcal{N}_L} \right],$$

and also let $b_i = \Delta P_i$. Based on the expression above, we can rewrite (13) as

$$b_i \approx A \left[\Psi_i^T \quad \Phi_i^T \right]^T. \tag{15}$$

Since (15) is an overdetermined system of equations, in the remainder of this section, we formulate the problem of computing $[\Psi_i^T, \Phi_i^T]^T$ in (15) as a TLS estimation problem. We note, however, that the ideas presented in this section are immediately applicable to estimate the unknown vectors in both systems described in (13)–(14).

A. Basic Total Least-Squares Approach

Before delving into the TLS estimation problem formulation and associated solution, we briefly describe the ordinary LSE problem formulation and its solution, as it applies to our setting. In ordinary LSE, since the regressor matrix is assumed to be error free, the rationale behind this estimation method is to correct the observations b_i as little as possible under the Euclidean norm metric; this can be formulated as an optimization program as follows (see, e.g., [20]):

$$\begin{aligned} &\min_{\hat{b}_i \in \mathbb{R}^M} \; ||\Delta b_i||_2 \,, \\ &\text{s.t.} \quad \hat{b}_i = A \left[\Psi_i^T \quad \Phi_i^T \right]^T, \end{aligned} \tag{16}$$

where $\Delta b_i = b_i - \hat{b}_i$. Once a minimizer, $\hat{b}_i$, is found, then any $[\hat{\Psi}_i^T, \hat{\Phi}_i^T]^T$ satisfying $\hat{b}_i = A[\hat{\Psi}_i^T, \hat{\Phi}_i^T]^T$ is a LSE solution to (15). We assume A has full column rank; under this condition, the closed-form unique solution to (16) is (see, e.g., [21])

$$\begin{bmatrix} \hat{\Psi}_i \\ \hat{\Phi}_i \end{bmatrix} = \left(A^T A \right)^{-1} A^T b_i. \tag{17}$$

In contrast to the LSE problem formulation in (16), since TLS estimation accounts for errors in A as well, analogous to the vector Euclidean norm, its problem formulation seeks to minimize the matrix Frobenius norm, as follows:

$$\begin{aligned} &\min_{[\hat{A} \ \hat{b}_i] \in \mathbb{R}^{M \times (\bar{N}+1)}} \quad ||[\Delta A \ \Delta b_i]||_F \,, \\ &\text{s.t.} \quad \hat{b}_i = \hat{A} \left[\Psi_i^T \quad \Phi_i^T \right]^T, \end{aligned} \tag{18}$$

where $\Delta A = A - \hat{A}$, $\Delta b_i = b_i - \hat{b}_i$, and $\bar{N} = N_G + 2N_L$ [20]. Then, once a minimizing $[\hat{A} \ \hat{b}_i]$ is found, then any $[\hat{\Psi}_i^T, \hat{\Phi}_i^T]^T$ satisfying $\hat{b}_i = \hat{A}[\hat{\Psi}_i^T, \hat{\Phi}_i^T]^T$ is a TLS solution to (15).

The solution to the TLS estimation problem in (18) relies heavily on the singular value decomposition (SVD) (see, e.g., [22]); below, we describe the procedure (see, e.g., [23])). To obtain the solution to (18), we rewrite (15) as (see, e.g., [20])

$$\left[A \quad b_i \right] \left[\Psi_i^T \quad \Phi_i^T \quad -1 \right]^T \approx 0. \tag{19}$$

By using the SVD, we can write

$$\left[A \quad b_i \right] = U \Sigma V^T, \tag{20}$$

where $U = [u_1, \ldots, u_M]$ and $V = [v_1, \ldots, v_{\bar{N}+1}]$ are unitary matrices, Σ is a diagonal matrix in which the diagonal elements σ_i are the singular values of $[A \ b_i]$ (see, e.g., [22]). If $\sigma_{\bar{N}+1} \neq 0$, then $[A \ b_i]$ has rank $\bar{N}+1$ and the unique solution to (19) is the zero vector. In order to obtain a nonzero solution to (19), the rank of $[A \ b_i]$ must be reduced to $\bar{N}$. According to the Eckart-Young-Mirsky low-rank matrix approximation theorem [24], the rank $\bar{N}$ approximation of $[A \ b_i]$, which minimizes the objective function in (18), is

$$\left[\hat{A} \quad \hat{b}_i \right] = U \hat{\Sigma} V^T, \tag{21}$$

where $\hat{\Sigma}$ is a diagonal matrix in which the diagonal elements $\hat{\sigma}_i = \sigma_i$, if $i < \bar{N} + 1$, and $\hat{\sigma}_i = 0$, otherwise. Since the approximate matrix $[\hat{A} \ \hat{b}_i]$ has rank $\bar{N}$, (19) has a nonzero solution. Based on properties of the SVD, $v_{\bar{N}+1}$ is the only vector that belongs to the null space of $[\hat{A}\ \hat{b}_i]$. Then, the TLS solution is obtained by scaling the vector $v_{\bar{N}+1}$ until its last component is equal to -1, namely,

$$\begin{bmatrix}\hat{\Psi}_i^T & \hat{\Phi}_i^T & -1\end{bmatrix}^T = -\frac{1}{v_{\bar{N}+1}^{\bar{N}+1}} v_{\bar{N}+1},$$

where $v_{\bar{N}+1}^{\bar{N}+1}$ denotes the $(\bar{N}+1)^{\text{th}}$ element of $v_{\bar{N}+1}$. Thus, the unique TLS solution to (15) is

$$\begin{bmatrix}\hat{\Psi}_i^T & \hat{\Phi}_i^T\end{bmatrix}^T = -\frac{1}{v_{\bar{N}+1}^{\bar{N}+1}} \begin{bmatrix}v_{\bar{N}+1}^1 & \cdots & v_{\bar{N}+1}^{\bar{N}}\end{bmatrix}^T. \tag{22}$$

Next, we illustrate the concepts introduced above.

Example 2 (3-Machine 9-Bus System): In this example, we consider the same system as in Example 1. Here, we use (22) to estimate the entries in each row of the power flow Jacobian matrix and compare the results to the benchmark values recorded in the left-half portion of Table I. In order to simulate PMU measurements of slight fluctuations in active and reactive power generated and consumed at each bus, we generate power injection (positive or negative) time-series data. To this end, we assume the active power injection at bus i at time instant k, denoted by $P_i[k]$, can be modeled as

$$P_i[k] = P_i^0[k] + P_i^0[k]\nu_1^P + \nu_2^P, \tag{23}$$

where $P_i^0[k]$ is the nominal active power injection at time instant k, and ν_1^P and ν_2^P are pseudorandom values drawn from standard normal distributions with zero mean and standard deviations $\sigma_1^P = 0.1$ and $\sigma_2^P = 0.1$, respectively. Similarly, we assume the reactive power injection at bus i at time instant k, denoted by $Q_i[k]$, can be modeled as

$$Q_i[k] = Q_i^0[k] + Q_i^0[k]\nu_1^Q + \nu_2^Q, \tag{24}$$

where $Q_i^0[k]$ is the nominal reactive power injection, and ν_1^Q and ν_2^Q are pseudorandom values drawn from standard normal distributions with zero mean and standard deviations $\sigma_1^Q = 0.1$ and $\sigma_2^Q = 0.1$, respectively. In both (23) and (24), there are two random components added to the deterministic nominal quantities. The first component, $P_i^0[k]\nu_1^P$ in (23) ($Q_i^0[k]\nu_1^Q$ in (24)), represents the inherent fluctuations in active (reactive) power generation and load. The second component, ν_2^P in (23) (ν_2^Q in (24)), represents random measurement noise, which is independent of the nominal active (reactive) power injection values. For each set of bus injection data, we solve the power flow equations, with the slack bus absorbing all power imbalances, to obtain the voltage magnitude and phase "measurements".

In this example, we simulate 100 sets of power injection and voltage measurements with the same network topology and operating point. In the right-half portion of Table I, we report TLS estimates corresponding to the entries in the left-half portion of Table I. By visually comparing the model-based sensitivities and measurement-based estimates in Table I, we note that the measurement-based TLS estimation achieves values that are very close to the model-based benchmark values obtained by directly linearizing the power flow equations. In addition to the visual comparison, we compute the mean-squared error (MSE) of each sensitivity vector and report them in Table II. In this case, the average MSE is 0.01169. ■

TABLE II: WECC 3-machine 9-bus systems—MSE of sensitivity factors obtained in Example 2 compared to corresponding model-based benchmark. To make the table more compact, all numerical quantities have been scaled up by $\times 10^3$.

$\hat{\Psi}_2$	0.09289	$\hat{\Phi}_2$	0.6921				
$\hat{\Psi}_3$	0.1357	$\hat{\Phi}_3$	0.9754				
$\hat{\Psi}_4$	13.73	$\hat{\Phi}_4$	22.28	$\hat{\Gamma}_4$	42.60	$\hat{\Lambda}_4$	58.39
$\hat{\Psi}_5$	8.616	$\hat{\Phi}_5$	41.82	$\hat{\Gamma}_5$	16.55	$\hat{\Lambda}_5$	5.933
$\hat{\Psi}_6$	7.446	$\hat{\Phi}_6$	11.53	$\hat{\Gamma}_6$	4.911	$\hat{\Lambda}_6$	22.12
$\hat{\Psi}_7$	2.792	$\hat{\Phi}_7$	6.452	$\hat{\Gamma}_7$	1.857	$\hat{\Lambda}_7$	11.84
$\hat{\Psi}_8$	10.26	$\hat{\Phi}_8$	1.054	$\hat{\Gamma}_8$	5.044	$\hat{\Lambda}_8$	1.219
$\hat{\Psi}_9$	2.987	$\hat{\Phi}_9$	7.142	$\hat{\Gamma}_9$	3.730	$\hat{\Lambda}_9$	2.831

B. Weighted Total Least-Squares Approach

One of the assumptions we make in (18) is that the Jacobian matrix sensitivity factors are approximately constant across the estimation time window. One way to eliminate this restriction and to obtain an estimator that is more adaptive to changes in operating point is to place more importance on recent measurements and less on earlier ones, which may be out of date. Again, before we delve into the WTLS estimation problem formulation, we briefly describe the ordinary weighted least-squares (WLS) estimation problem setting in which the objective function in (16) becomes

$$\min_{\hat{b}_i \in \mathbb{R}^M} \quad \left|\left|\sqrt{W}\Delta b_i\right|\right|_2, \tag{25}$$

where W is a positive definite symmetric matrix. The solution to (25) is given by (see, e.g., [21])

$$\begin{bmatrix}\hat{\Psi}_i \\ \hat{\Phi}_i\end{bmatrix} = \left(A^T W A\right)^{-1} A^T W b_i. \tag{26}$$

The idea is to choose appropriate values for W so that more recent measurements are weighted preferentially over past ones. If the elements of the error vector Δb_i are uncorrelated, then $W = [W(i,j)]$ is a diagonal matrix. The WLS estimation problem is often formulated using an exponential forgetting factor [25], in which the more recent measurements are preferentially weighted by setting $W(i,i) = f^{M-i}$ for some fixed $f \in (0,1]$, where f is called a "forgetting" factor.

In the WTLS estimation problem setting, the optimization in (18) becomes

$$\begin{aligned} &\min_{[\hat{A}\ \hat{b}_i] \in \mathbb{R}^{M \times (\bar{N}+1)}} F_0\left(\Delta A, \Delta b_i\right), \\ &\text{s.t.} \qquad \hat{b}_i = \hat{A}\begin{bmatrix}\Psi_i^T & \Phi_i^T\end{bmatrix}^T, \end{aligned} \tag{27}$$

with

$$F_0(\cdot) = \sum_{k=1}^{M} \Delta a[k] W_k \Delta a[k]^T + w_k \Delta b_i[k]^2, \tag{28}$$

where $\Delta a[k]$ denotes the k^{th} row of ΔA, $\Delta b_i[k]$ is the k^{th} element of Δb_i, and matrix W_k and scalar w_k represent weighting factors for elements in $\Delta a[k]$ and $\Delta b_i[k]$, respectively. Next, we discuss the selection of these weighting factors.

1) Choice of Weighting Factors: Inspired by ordinary WLS estimation, we set $w_k = f^{M-k}$, so as to weigh the more recent elements in the observation vector, b_i, more heavily. With regard to the choice of W_k's, first, we assume that the elements of the error vector $\Delta a[k]$ are uncorrelated; therefore the matrix

$W_k = [W_k(i,j)]$ is diagonal. Furthermore, if measurements obtained at each bus are equally reliable, then the elements of $\Delta a[k]$ are equally weighted. Then, by employing the exponential forgetting factor, we set $W_k(i,i) = w_k$, for all i. With the above choices for w_k and W_k, if $f = 1$, then all measurements are given equal weighting, and the WTLS formulation in (27) is equivalent to the TLS one in (18). On the other hand, if $f < 1$, then earlier measurements would not contribute as much to the final estimate $[\hat{\Psi}_i^T, \hat{\Phi}_i^T]^T$ as more recent ones. In this way, the WTLS formulation is useful if the system experiences a change in operating point during the measurement acquisition time window. With the weighting factors chosen as described above, we next describe the solution to the optimization problem in (27).

2) WTLS Problem Solution: Note that if $W_k = \mathbb{I}_{\bar{N}}$, where $\mathbb{I}_{\bar{N}}$ denotes an $\bar{N} \times \bar{N}$ identity matrix, and $w_k = 1$, for all $k = 1, \ldots, M$, then the formulation in (27) is equivalent to that in (18). Unlike the basic TLS problem, however, the WTLS problem does not have a SVD-based closed-form solution. In order to solve (27), we follow the development described in [26], which is summarized below. We first note that the equality constraint in (27) is equivalent to $b_i - \Delta b_i = (A - \Delta A)[\hat{\Psi}_i^T, \hat{\Phi}_i^T]^T$, i.e.,

$$\Delta b_i[k] = \begin{bmatrix} \Psi_i^T & \Phi_i^T \end{bmatrix} \left(\Delta a[k]^T - a[k]^T \right) + b_i[k], \tag{29}$$

for each $k = 1, \ldots, M$, where $a[k]$ denotes the k^{th} row of A. Substituting (29) into (28), we obtain the following unconstrained optimization problem:

$$\min_{\hat{A}, [\Psi_i^T, \Phi_i^T]^T} F_u \left(\begin{bmatrix} \Psi_i^T & \Phi_i^T \end{bmatrix}^T, \Delta A \right), \tag{30}$$

where

$$F_u(\cdot) = \sum_{k=1}^{M} \Delta a[k] W_k \Delta a[k]^T + w_k \Big(\begin{bmatrix} \Psi_i^T & \Phi_i^T \end{bmatrix} \Delta a[k]^T - \begin{bmatrix} \Psi_i^T & \Phi_i^T \end{bmatrix} a[k]^T + b_i[k] \Big)^2. \tag{31}$$

We note that $F_u(\cdot)$ is differentiable with respect to $\Delta a[k]$, for each $k = 1, \ldots, M$. Suppose ΔA^* is a local minimizer of (30). Then, according to first-order necessary conditions for optimality, at ΔA^* (see, e.g., [27, Chap. 11]), we have that

$$0 = \left. \frac{dF_u}{d\Delta a[k]} \right|_{\Delta a[k] = \Delta a^*[k]}, \; k = 1, \ldots, M,$$

from which we obtain

$$\Delta a^*[k]^T = \left[W_k + \begin{bmatrix} \Psi_i \\ \Phi_i \end{bmatrix} \begin{bmatrix} \Psi_i^T & \Phi_i^T \end{bmatrix} w_k \right]^{-1} \times w_k \left(\begin{bmatrix} \Psi_i^T & \Phi_i^T \end{bmatrix} a[k]^T - b_i[k] \right) \begin{bmatrix} \Psi_i \\ \Phi_i \end{bmatrix}, \tag{32}$$

for each $k = 1, \ldots, M$. By invoking the matrix inversion lemma (see, e.g., [22]), (32) simplifies to

$$\Delta a^*[k]^T = \frac{\begin{bmatrix} \Psi_i^T & \Phi_i^T \end{bmatrix} a[k]^T - b_i[k]}{w_k^{-1} + \begin{bmatrix} \Psi_i^T & \Phi_i^T \end{bmatrix} W_k^{-1} \begin{bmatrix} \Psi_i \\ \Phi_i \end{bmatrix}} W_k^{-1} \begin{bmatrix} \Psi_i \\ \Phi_i \end{bmatrix}. \tag{33}$$

Finally, we substitute each optimal $\Delta a^*[k]$ as given in (33) into (31), from which we reformulate the optimization problem in (30) as

$$\min_{[\Psi_i^T, \Phi_i^T]^T} F \left(\begin{bmatrix} \Psi_i^T & \Phi_i^T \end{bmatrix}^T \right), \tag{34}$$

where

$$F(\cdot) = \sum_{k=1}^{M} \frac{\left(\begin{bmatrix} \Psi_i^T & \Phi_i^T \end{bmatrix} a[k]^T - b_i[k] \right)^2}{w_k^{-1} + \begin{bmatrix} \Psi_i^T & \Phi_i^T \end{bmatrix} W_k^{-1} \begin{bmatrix} \Psi_i \\ \Phi_i \end{bmatrix}}. \tag{35}$$

Through the development above, we convert the original constrained WTLS problem in (27) into the less troublesome unconstrained minimization problem in (34). The optimization problem in (27) (or (34)) is nonconvex; therefore, numerical solution methods do not guarantee convergence to a global minimum. Many numerical algorithms, most of which are iterative, have been proposed to solve (27) or (34) (see, e.g., [23], for an overview). For our case studies, we find that built-in optimization routines in MATLAB are sufficient as proof-of-concept to demonstrate the feasibility of the proposed power flow Jacobian estimation framework. In a commercial implementation of the proposed framework, it may be prudent to investigate convergence properties of various solution methods. We refrain from further discussion on this topic here as it is beyond the scope of the present work. Next, we illustrate the ideas presented above with an example.

Example 3 (3-Machine 9-Bus System): We consider the same system as in Example 1 and simulate 200 sets of PMU measurements of slight fluctuations. In order to simulate an undetected change in operating point, without updating the model, the active load at bus 6 linearly increases by 1.6 p.u. over the span of 20 measurements beginning at $k = 80$, with the generation at bus 2 also increasing commensurately at each time step.

As in Example 2, we compute the power flow, with the slack bus absorbing all power imbalances for each particular time k. Table III shows a comparison between benchmark sensitivity factors obtained via direct linearization of the power flow equations around the operating point (both before and after the change), and those obtained via the proposed WTLS framework with forgetting factors $f = 0.96$ and $f = 1$. Both measurement-based estimations are executed at $k = 200$ with the previous $M = 200$ measurements. Since the operating point is undetected by operators, under the pre-change system model, the power flow Jacobian matrix (some entries of which are shown in column 3 of Table III) results in an average MSE of 0.2956. In column 5 of Table III, we record results for WTLS with $f = 1$ (or, equivalently, TLS). From the average MSE metric, as well as a survey of the individual values, reported in this column, we note that the basic TLS scheme is unable to estimate the updated Jacobian matrix elements. On the other hand, the WTLS method with $f = 0.96$ is able to track elements in the Jacobian matrix, as shown in column 4 of Table III. However, compared to the SVD computation in basic TLS, the WTLS optimization incurs much higher computational burden; thus, the cost of better tracking is longer computation time and lack of optimality guarantee.

TABLE III: WECC 3-machine 9-bus systems—model- and measurement-based sensitivity factors obtained in Example 3.

	Model-based		Measurement-based	
	Post-change	Pre-change	$f = 0.96$	$f = 1$
Ψ_5^2	0	0	-0.5254	-0.2637
Ψ_5^3	0	0	0.08467	0.9688
Ψ_5^4	-9.685	-10.86	-9.599	-10.66
Ψ_5^5	0	0	-0.2559	0.1843
Ψ_5^6	14.60	16.54	14.97	15.52
Ψ_5^7	0	0	0.6563	0.2575
Ψ_5^8	0	0	-0.1905	0.4154
Ψ_5^9	-4.911	-5.6816	-5.137	-6.780
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
	Average MSE	0.2956	0.1979	0.4501

By observing estimation results in both Tables I and III, we note that while the TLS-based schemes are able to track the nonzero terms with sufficient accuracy, the resulting estimated signal is quite noisy, with many near-zero terms. ■

IV. Estimation with a Subset of Measurements

In the proposed Jacobian matrix estimation framework presented thus far, to estimate the unknown sensitivity factors, with respect to bus i, voltage magnitude and phase angle measurements are required from all buses. In other words, the framework necessitates a central data collector to whom all measurements are passed. Moreover, it requires at least as many time-sampled sets of measurements as the number of columns of the Jacobian matrix. Both of these restrictions become unwieldy for large-scale power systems. First, since power systems are constantly undergoing changes and operators often need to quickly determine the current system state, it would be ideal to obtain accurate estimates using fewer data sets. Furthermore, in a practical setting, the entire set of measurements may not be available for transmission to a central data collector.

In order to relax the restrictions described above, we note that, due to the structure of the power flow equations, the sensitivity factors Ψ_i^j, Φ_i^j, Γ_i^j, and Λ_i^j are only nonzero if $i = j$, or if there exists a transmission line connecting buses i and j. Such information can be obtained a priori from a known base-case system network model or from real-time breaker status information. With this in mind, we define $\mathcal{N}_i$ as the set of buses that are connected to bus i, including bus i itself. Then, based on the full systems of equations in (13)–(14), we can obtain the following reduced systems of equations:

$$\Delta P_i \approx \left[(\Delta\theta_j)_{j\in(\mathcal{N}_G\cup\mathcal{N}_L)\cap\mathcal{N}_i} \;\; (\Delta V_j)_{j\in\mathcal{N}_L\cap\mathcal{N}_i} \right] \begin{bmatrix} \Psi_i^{\mathcal{N}_i} \\ \Phi_i^{\mathcal{N}_i} \end{bmatrix}, \tag{36}$$

where

$$\Psi_i^{\mathcal{N}_i} = \left[\left(\Psi_i^j \right)_{j\in(\mathcal{N}_G\cup\mathcal{N}_L)\cap\mathcal{N}_i} \right] \text{ and } \Phi_i^{\mathcal{N}_i} = \left[\left(\Phi_i^j \right)_{j\in\mathcal{N}_L\cap\mathcal{N}_i} \right]$$

are reduced sensitivity vectors that contain only the nonzero entries of Ψ_i and Φ_i, respectively; and

$$\Delta Q_i \approx \left[(\Delta\theta_j)_{j\in(\mathcal{N}_G\cup\mathcal{N}_L)\cap\mathcal{N}_i} \;\; (\Delta V_j)_{j\in\mathcal{N}_L\cap\mathcal{N}_i} \right] \begin{bmatrix} \Gamma_i^{\mathcal{N}_i} \\ \Lambda_i^{\mathcal{N}_i} \end{bmatrix}, \tag{37}$$

where

$$\Gamma_i^{\mathcal{N}_i} = \left[\left(\Gamma_i^j \right)_{j\in(\mathcal{N}_G\cup\mathcal{N}_L)\cap\mathcal{N}_i} \right] \text{ and } \Lambda_i^{\mathcal{N}_i} = \left[\left(\Lambda_i^j \right)_{j\in\mathcal{N}_L\cap\mathcal{N}_i} \right]$$

contain only the nonzero entries of Γ_i and Λ_i, respectively.

Similar to the full formulation in (13)–(14), we can obtain estimates of the reduced sensitivity vectors in (36)–(37) via (22) and the solution of (34). Unlike the full formulation, however, to obtain estimates of these reduced sensitivity factors with respect to bus i, it suffices to acquire $M > 2(\#\mathcal{N}_i)$ sets of synchronized measurements to compute a row of the Jacobian matrix,[1] thus reducing the computational burden involved.

The solution to (36)–(37) can be computed at each bus, thus enabling parallel processing, so that the full system topology and relevant parameters can be obtained quickly. The local topology information can be transmitted to a central controller periodically, or when the resulting estimates indicate an update is required.

V. Case Studies

We use the proposed measurement-based approach to estimate the Jacobian matrix in the IEEE 118-bus system. The simulation tool MATPOWER [19] is used throughout to solve the power flow and generate voltage magnitude and phase angle measurements from pseudo-random bus injections generated using (23) and (24).

A. Base Case

We consider the base case model for the IEEE 118-bus system and assess the effectiveness of the proposed measurement-based method to estimate the power flow Jacobian matrix under constant nominal operating point. As in Example 2, we simulate bus injection data by adding noise to the nominal injections, as given in (23)–(24), with $\sigma_1^P = \sigma_1^Q = 0.03$ and $\sigma_2^P = \sigma_2^Q = 0.01$. For comparison, we obtain benchmark values by linearizing the power flow equations around the nominal operating point.

1) Measurements from All Buses: We utilize data from all buses and compute estimates for the elements of the power flow Jacobian by solving the full problems in (13)–(14). We assume the time window under consideration contains $M = 1000$ sets of synchronized measurements. Using (22), in conjunction with simulated measurements from all buses, we obtain estimates of Ψ_i and Φ_i, for $i \in \mathcal{N}_G \cup \mathcal{N}_L$, as well as Γ_i and Λ_i, for $i \in \mathcal{N}_L$. When comparing these estimated vectors to their corresponding model-based benchmark values, we find that the mean MSE for all estimated vectors is 0.00497, with the maximum being 0.5090.

2) Measurements from a Subset of Buses: Suppose each bus is equipped with the computational capability required to conduct its own sensitivity estimation. Then, as described in Section IV, each estimation problem solves fewer unknown sensitivity factors and requires fewer sets of synchronized measurements. Therefore, we use the first $M = 40$ sets of

[1] $\#\mathcal{A}$ denotes the cardinality of set $\mathcal{A}$.

measurements from the full-system Jacobian matrix estimation above. We assume that each bus is able to attain voltage magnitude and phase angle measurements from its neighbors. Via (34), we solve for the unknown vectors in (36)–(37) for each $i = 1, \ldots, N$, and further compare them to corresponding model-based benchmark values. We find the average MSE to be 0.001523, with the maximum being 0.1936.

B. Change in Topology

Under the reduced formulation presented in Section IV, we assess the performance of the proposed WTLS framework, as described in Section III-B, to update the entries of the Jacobian matrix after a topology change. With respect to this, we simulate $M = 100$ sets of synchronous measurements by computing the power flow solution using power injection data generated via (23)–(24) with $\sigma_1^P = \sigma_1^Q = \sigma_2^P = \sigma_2^Q = 0.1$. To simulate a topology change, we introduce a credible line outage (i.e., one that does not island the system) at time step $k = 30$. As in Example 3, we use a forgetting factor of $f = 0.96$. We repeatedly simulate random sample paths with random line outages.

Overall, the proposed WTLS estimation method is able to adapt and obtain accurate estimates for 63.84% of the affected Jacobian matrix entries, where the estimate is deemed "accurate" if it is within 10% of the corresponding post-change Jacobian entry. Since the optimization problem in WTLS is nonconvex, iterative numerical solution methods may only attain a local minimum, as evidenced by the low estimation accuracy. In contrast, for the same random sample paths and forgetting factor, the WLS estimates, obtained via (26), are accurate for 84.72% of the affected entries.

VI. Concluding Remarks

In this paper, we presented a measurement-based method to estimate the power flow Jacobian matrix without relying on the system power flow model. The proposed method relies on the solution of an overdetermined set of linear equations constructed from real-time measurements obtained with PMUs installed throughout the system. Via TLS estimation, we account for measurement errors in both the observation vector as well as the regressor matrix. We showed that the proposed method provides accurate estimates of the Jacobian matrix entries. Furthermore, we improve the adaptability of the proposed method by employing WTLS and WLS estimation.

As future work, we would like to take advantage of the sparsity structure in the Jacobian matrix to obtain estimates without relying on a priori base-case topology or real-time breaker status information. Moreover, the method described in Section IV motivates further exploration into coordinated Jacobian-matrix estimation within a hierarchical computation architecture.

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