Power-flow Sensitivities in DC Distribution Systems

Victor Purba
Department of Electrical and Computer Engineering
University of Minnesota
Minneapolis, Minnesota 55455
Email: purba002@umn.edu

Yu Christine Chen
Department of Electrical and Computer Engineering
The University of British Columbia
Vancouver, British Columbia V6T 1Z4
Email: chen@ece.ubc.ca

Sairaj V. Dhople
Department of Electrical and Computer Engineering
University of Minnesota
Minneapolis, Minnesota 55455
Email: sdhople@umn.edu

***Abstract*—We examine the steady-state behavior of DC distribution systems with arbitrary topologies that couple a heterogeneous collection of power-electronic circuits modeled as parallel connections of resistances, and constant current and power sources (or loads). For this system, we provide analytical closed-form expressions for sensitivities that capture variations of line power flows due to: i) changes in nodal power injections, and ii) line outages. Our results follow from a perturbative analysis of the network power-balance expressions, and the approach leverages contemporary results from circuit theory and structural properties of the network conductance matrix. Simulations from illustrative networks confirm the validity and accuracy of the analytical expressions in estimating the impact of nodal power injections and line outages.**

I. Introduction

This paper provides analytical expressions for sensitivities of line power flows in DC distribution systems with respect to changes in nodal power injections and line outages. The analysis applies to DC networks that consist of a heterogeneous collection of power-electronic circuits modeled as parallel connections of resistances, and constant current and power sources (or loads) operating in steady state. The network topology is assumed to be connected and composed of resistive interconnections. The analysis framework is intended to outline strategies for online static security assessment of complex DC distribution networks.

Advancements in semiconductor technologies, materials, and distributed control of switching power-electronic circuits have propelled DC distribution systems to the forefront in applications including (but not limited to) HVDC transmission systems, shipboard systems, telecommunication systems, and microgrids [1], [2]. Advantages of DC distribution networks such as higher power transfer capability, better wire utilization, and improved system efficiency, are particularly appreciable in islanded microgrid applications involving DC sources such as photovoltaic panels and fuel cells [3]. Research in the domain of DC distribution systems has dominantly focused on the modeling, analysis, and control of the individual power-electronic circuits. System-theoretic frameworks to study the networked behaviors of collections of power-electronic circuits are only recently receiving attention [4], [5]. It is worth mentioning that networked behaviors of synchronous generators in the bulk AC power system have been investigated from a variety of standpoints [6]. With particular reference to the methods proposed in this work, security assessment tools that indicate the ability of the system to withstand disturbances have received significant attention in AC power networks [7].

We expect the sensitivity analysis framework outlined here to be critical in online static security-assessment tools to monitor and maintain the operational reliability of complex DC distribution networks. To illustrate the relevance of analytically grounded approaches in such applications, consider the ubiquitous $N-1$ contingency analysis, which quantifies the capability of the system to meet operational reliability requirements in case of an outage in a particular asset [8]. For a large network with many contingencies to consider, repeated simulations to quantify operational reliability are computationally expensive, and do not yield any theoretical guarantees. An alternative is to use an estimate of the current operating point together with linear sensitivities—such as the ones we derive in this work—to approximate the change in power flows in the network to changes in nodal injections or line outages.

The derivation of the sensitivities proceeds from the analysis of a matrix-vector representation of power balance equations. We also leverage contemporary results from circuit theory, particularly, structural properties of the network conductance matrix. Consequently, the expressions for the sensitivities reflect the network topology, as well as the operating point (i.e., the steady-state voltages). For AC power systems, analogous sensitivities have been used extensively to verify system operational reliability in steady-state operation. Of particular relevance and related to the sensitivities that we analyze here are the so-called injection shift factors and line outage distribution factors (see, e.g., [9]–[12]).

The remainder of this paper is organized as follows. In Section II, we establish notation, provide a few mathematical preliminaries, and describe the electrical network model. In Section III, we derive the current and power injection sensitivities to bus injections and line outages. Case studies are presented to validate the analysis for illustrative networks in Section IV. To demonstrate the accuracy of our expressions, we compare results with those recovered by solving the nonlinear power-balance equations. Finally, we conclude this paper in Section V by outlining a few pertinent directions for future work.

II. Preliminaries

A. Notation and Mathematical Preliminaries

The matrix transpose will be denoted by $(\cdot)^{\mathrm{T}}$. The spaces of $N \times 1$ real-valued vectors is denoted by $\mathbb{R}^N$; and $\mathbb{R}^{N\times N}$ denotes the space of $N \times N$ real-valued matrices. A diagonal matrix formed with diagonal entries composed of entries of the vector x is denoted by $\mathrm{diag}(x)$; and $\mathrm{diag}(x/y)$ forms a diagonal matrix with the ℓth entry given by x_ℓ/y_ℓ, where x_ℓ and y_ℓ are the ℓth entries of vectors x and y, respectively. The vector recovered from the entry-wise product of two vectors, x and y, is denoted by $x \circ y$; and x^2 denotes $x \circ x$. The $N \times N$ identity matrix is denoted by I_N; $\mathbb{1}_{M\times N}$ denotes the $M \times N$ matrices with all ones; $\mathbb{1}_N$ and $\mathbb{0}_N$ denote $N \times 1$ column vectors of all ones and zeros, respectively; e_j denotes a column vector of all zeros except with the jth entry equal to 1; and $e_{j\ell}$ denotes a column vector of all zeros except with the jth and ℓth entries equal to 1 and -1, respectively. The (j,ℓ)th entry of matrix X is denoted by $[X]_{j\ell}$. The set $\mathbb{1}_N^\perp$ denotes the subspace of all vectors orthogonal to the span of $\mathbb{1}_N$ in $\mathbb{R}^N$.

A matrix $X \in \mathbb{R}^{N\times N}$ is irreducibly diagonally dominant if [13]: i) X is irreducible; ii) X is diagonally dominant, i.e., $|[X]_{jj}| \geq \sum_{\ell\neq j} |[X]_{j\ell}|$, $\forall\, j = 1,\dots,N$; and iii) for some j, it holds that $|[X]_{jj}| > \sum_{\ell\neq j} |[X]_{j\ell}|$. Irreducibly diagonally dominant matrices are invertible [13].

B. Network description

Consider a DC electrical network with N nodes operating in steady state. Each node may be connected to a shunt element composed of the parallel connection of (a subset of) a resistive element, a constant current source/load, and a constant power source/load. The nodes of the network are collected in the set $\mathcal{N} = \{1,\dots,N\}$, and branches (edges) are collected in the set $\mathcal{E} := \{(j,\ell)\} \subseteq \mathcal{N}\times\mathcal{N}$. Denote the vectors that collect the nodal current injections, voltages, and power injections in the network by $i = [i_1,\dots,i_N]^{\mathrm{T}} \in \mathbb{R}^N$, $v = [v_1,\dots,v_N]^{\mathrm{T}} \in \mathbb{R}^N$, and $p = [p_1,\dots,p_N]^{\mathrm{T}} \in \mathbb{R}^N$, respectively. The circuit laws that capture Kirchhoff's current law and Ohm's law in the network can be expressed in matrix-vector form as

$$i = Gv, \tag{1}$$

where $G \in \mathbb{R}^{N\times N}$ is the conductance matrix with entries specified as

$$[G]_{j\ell} := \begin{cases} g_j + \sum_{(j,k)\in\mathcal{E}} g_{jk}, & \text{if } j = \ell, \\ -g_{j\ell}, & \text{if } (j,\ell)\in\mathcal{E}, \\ 0, & \text{otherwise}, \end{cases} \tag{2}$$

with $g_j \in \mathbb{R}_{\geq 0}$ denoting the shunt conductance at node j, and $g_{j\ell} = g_{\ell j} \in \mathbb{R}_{\geq 0}$ the conductance of the line (j,ℓ). For a connected electrical network, the presence of a single shunt resistive element guarantees invertibility of G by inducing irreducible diagonal dominance (see Section II-A, and also [13]). Power balance in the network can be written in the following compact matrix-vector form:

$$p = \mathrm{diag}(v) i = \mathrm{diag}(v) G v, \tag{3}$$

where the second equality above follows from substituting for the current injections from (1).

III. Sensitivities of Current and Power Flows

We are interested in the power-injection sensitivities, by which we mean, the sensitivities of the line power flows to nodal power injections. To this end, we begin with a derivation of the current-injection sensitivities in Section III-A, by which we mean the sensitivities of the line current flows to nodal current injections. We then derive the power-injection sensitivities in Section III-B, and subsequently expressions for sensitivities of power flows to line outages in Section III-C.

A. Current Injection Sensitivities

The current injected into the (j,ℓ) line (measured at node j) is given by

$$i_{(j,\ell)} = (v_j - v_\ell)\, g_{j\ell} + v_j g_j, \tag{4}$$

where $g_{j\ell}$ is the conductance of the (j,ℓ) line, and g_j is the shunt conductance of the j node. Since we are interested in relating the line current flows to the nodal current injections, we write (4) as

$$\begin{aligned} i_{(j,\ell)} &= (g_{j\ell} e_{j\ell}^{\mathrm{T}} + g_j e_j^{\mathrm{T}}) v \\ &= (g_{j\ell} e_{j\ell}^{\mathrm{T}} + g_j e_j^{\mathrm{T}}) G^{-1} i =: \kappa_{(j,\ell)}^{\mathrm{T}} i, \end{aligned} \tag{5}$$

where in the second equality above, we substitute for the nodal voltages by inverting (1). It follows that the sensitivity of the (j,ℓ) line current to the nodal current injections is captured by

$$\Delta i_{(j,\ell)} = \kappa_{(j,\ell)}^{\mathrm{T}} \Delta i, \tag{6}$$

where the *current injection sensitivity vector*, $\kappa_{(j,\ell)} \in \mathbb{R}^N$, is specified by

$$\kappa_{(j,\ell)}^{\mathrm{T}} = (g_{j\ell} e_{j\ell}^{\mathrm{T}} + g_j e_j^{\mathrm{T}}) G^{-1}. \tag{7}$$

Notice that the entries of $\kappa_{(j,\ell)}$ are independent of i, v, and p, and they only depend on the topology and constitution of the electrical network. In short, the sensitivity of the line flows to the current injections are independent of the operating point of the network.

We also consider the case when the DC distribution system has no shunt conductance elements, i.e., $g_j = 0, \forall\, j \in \mathcal{N}$. Consequently, it follows that the conductance matrix, G, is non-invertible, with the null space consisting of the vector $\mathbb{1}_N$. In order to satisfy Kirchhoff's current law and Ohm's law captured in (1), we restrict the nodal current injection vector, i, to be in the vector space of $\mathbb{1}_N^\perp$. In this case, the current through line (j,ℓ) is

$$i_{(j,\ell)} = g_{j\ell}(v_j - v_\ell) = g_{j\ell} e_{j\ell}^{\mathrm{T}} v. \tag{8}$$

Furthermore, the nodal voltages satisfy

$$v = G^\dagger i + \frac{1}{N}\mathbb{1}_N \mathbb{1}_N^{\mathrm{T}} v, \tag{9}$$

which follows from pre-multiplying both sides of (1) by the pseudoinverse of the conductance matrix, $G^{\dagger}$, and recognizing that G and $G^{\dagger}$ are related by

$$GG^{\dagger} = G^{\dagger}G = I_N - \frac{1}{N}\mathbb{1}_{N\times N}. \tag{10}$$

Substituting for v from (9) in (8), and utilizing the simple fact $e_{j\ell}^{\mathrm{T}}\mathbb{1}_{N\times N} = \mathbb{0}_N^{\mathrm{T}}$, we see that the current-injection sensitivity vector in this case is specified as

$$\kappa_{(j,\ell)}^{\mathrm{T}} = g_{j\ell}e_{j\ell}^{\mathrm{T}}G^{\dagger}. \tag{11}$$

B. Sensitivities of Line Flows to Bus Injections

In this section, we derive closed-form expression for the sensitivities of line power flows to variations in nodal power injections. We start with the first-order sensitivity of power injections to variations in terminal voltages in the DC network, captured by $\Delta p \approx J\Delta v$, where $\Delta v = [\Delta v_1, \dots, \Delta v_N]^{\mathrm{T}}$ is the vector that denotes variations in nodal voltages, $\Delta p = [\Delta p_1, \dots, \Delta p_N]^{\mathrm{T}}$ denotes the vector that captures variations in nodal power injections, and the Jacobian of the power flow equations, J, is given by

$$J := \mathrm{diag}(v)G + \mathrm{diag}(Gv). \tag{12}$$

This expression is obtained from the matrix-vector representation of the power injections in the network, $p = \mathrm{diag}(v)Gv$ (see (3)), as follows:

$$\begin{aligned}\Delta p &\approx \mathrm{diag}\,(\Delta v)\,Gv + \mathrm{diag}\,(v)\,G\Delta v\\ &= (\mathrm{diag}\,(Gv) + \mathrm{diag}\,(v)\,G)\,\Delta v =: J\Delta v,\end{aligned} \tag{13}$$

where the second equality above follows from the fact that for two N dimensional vectors, x and y, $\mathrm{diag}\,(x)\,y = x \circ y = y \circ x = \mathrm{diag}\,(y)\,x$.

Remark. (Invertibility of J) For $v, i \neq \mathbb{0}_N$, and $i_j \geq -v_j g_j, \forall\, j \in \mathcal{N}$, the Jacobian, J, is invertible. To show this, first note that for networks with shunt conductance elements, G is invertible since it is irreducibly diagonally dominant. For v not identically equal to zero, it follows that $\mathrm{diag}\,(v)\,G$ is irreducibly diagonally dominant. Finally, for i not identically equal to zero, and $i_j \geq -v_j g_j$ for all $j \in \mathcal{N}$, it follows that $\mathrm{diag}\,(i) + \mathrm{diag}\,(v)\,G = \mathrm{diag}\,(Gv) + \mathrm{diag}\,(v)\,G = J$ is irreducibly diagonally dominant and hence, invertible.

With the sensitivities of nodal power injections and voltages quantified, we next move to the power-injection sensitivities. Denote the power flow on the (j,ℓ) line, measured at node j, by $p_{(j,\ell)}$. This can be expressed as:

$$\begin{aligned}p_{(j,\ell)} &= v_j \cdot i_{(j,\ell)}\\ &= \left(e_j^{\mathrm{T}}v\right)\left(\kappa_{(j,\ell)}^{\mathrm{T}}i\right) = e_j^{\mathrm{T}}v\kappa_{(j,\ell)}^{\mathrm{T}}\mathrm{diag}\left(\frac{\mathbb{1}_N}{v}\right)p,\end{aligned} \tag{14}$$

where in the second equality above, we substitute for the line current $i_{(j,\ell)}$ from (5), and in the third equality we substitute for the terminal current-injection vector in terms of the power injections from (3). The first-order sensitivities of the power flows can then be obtained as:

$$\begin{aligned}\Delta p_{(j,\ell)} &\approx e_j^{\mathrm{T}}v\kappa_{(j,\ell)}^{\mathrm{T}}\,\mathrm{diag}\left(\frac{\mathbb{1}_N}{v}\right)\Delta p\\ &\quad + e_j^{\mathrm{T}}J^{-1}\Delta p\kappa_{(j,\ell)}^{\mathrm{T}}\,\mathrm{diag}\left(\frac{\mathbb{1}_N}{v}\right)p\\ &\quad - e_j^{\mathrm{T}}v\kappa_{(j,\ell)}^{\mathrm{T}}\,\mathrm{diag}\left(\frac{J^{-1}\Delta p}{v^2}\right)p,\end{aligned} \tag{15}$$

where we have substituted $\Delta v = J^{-1}\Delta p$ from (13). Since $e_j^{\mathrm{T}}J^{-1}\Delta p$ and $\kappa_{(j,\ell)}^{\mathrm{T}}\mathrm{diag}\,(1/v)\,p$ are both scalar quantities, it follows that the second term in (15) can be written as

$$\begin{aligned}&e_j^{\mathrm{T}}J^{-1}\Delta p\kappa_{(j,\ell)}^{\mathrm{T}}\,\mathrm{diag}\left(\frac{\mathbb{1}_N}{v}\right)p\\ &= \kappa_{(j,\ell)}^{\mathrm{T}}\,\mathrm{diag}\left(\frac{\mathbb{1}_N}{v}\right)pe_j^{\mathrm{T}}J^{-1}\Delta p.\end{aligned} \tag{16}$$

Rearranging quantities in the third term in (15), we get

$$\begin{aligned}&e_j^{\mathrm{T}}v\kappa_{(j,\ell)}^{\mathrm{T}}\,\mathrm{diag}\left(\frac{J^{-1}\Delta p}{v^2}\right)p\\ &= e_j^{\mathrm{T}}v\kappa_{(j,\ell)}^{\mathrm{T}}\,\mathrm{diag}\left(\frac{\mathbb{1}_N}{v^2}\right)\mathrm{diag}\left(J^{-1}\Delta p\right)p\\ &= e_j^{\mathrm{T}}v\kappa_{(j,\ell)}^{\mathrm{T}}\,\mathrm{diag}\left(\frac{p}{v^2}\right)J^{-1}\Delta p,\end{aligned} \tag{17}$$

where the second equality above follows from $\mathrm{diag}\,(x)\,y = \mathrm{diag}\,(y)\,x$. Finally, substituting (16) and (17) into (15) and suitably rearranging and collecting terms, we see that, to first order, variation in the power-flow on the (j,ℓ) line, $\Delta p_{(j,\ell)}$, induced due to variations in nodal power injections, Δp, can be expressed as

$$\Delta p_{(j,\ell)} \approx \rho_{(j,\ell)}^{\mathrm{T}}\Delta p, \tag{18}$$

where, $\rho_{(j,\ell)}$ is given by

$$\begin{aligned}\rho_{(j,\ell)}^{\mathrm{T}} &= e_j^{\mathrm{T}}v\kappa_{(j,\ell)}^{\mathrm{T}}\left(\mathrm{diag}\left(\frac{\mathbb{1}_N}{v}\right) - \mathrm{diag}\left(\frac{p}{v^2}\right)J^{-1}\right)\\ &\quad + \kappa_{(j,\ell)}^{\mathrm{T}}\,\mathrm{diag}\left(\frac{\mathbb{1}_N}{v}\right)pe_j^{\mathrm{T}}J^{-1},\end{aligned} \tag{19}$$

with $v \in \mathbb{R}^N$ and $p \in \mathbb{R}^N$ denoting the nominal terminal voltage and power injections in the network, respectively; $\kappa_{(j,\ell)}$ denoting the sensitivities of current flows on the (j,ℓ) line (see (6)); and J is the Jacobian of the power-flow equations defined in (12).

Remark. (Flat voltage profile) Suppose the voltage profile across the network is flat, i.e., $v_\ell = v_j\,\forall\,\ell, j \in \mathcal{N}$, and furthermore, suppose that the nodal voltages are fixed to their nominal values (e.g., through local feedback control), following which, $\Delta v \approx 0$. From (15), neglecting the terms that are multiplied by Δv, we get the following simplified expression for the power-injection sensitivities:

$$\begin{aligned}\Delta p_{(j,\ell)} &\approx e_j^{\mathrm{T}}v\kappa_{(j,\ell)}^{\mathrm{T}}\,\mathrm{diag}\left(\frac{\mathbb{1}_N}{v}\right)\Delta p\\ &= \kappa_{(j,\ell)}^{\mathrm{T}}\,\mathrm{diag}\left(\frac{v_j\mathbb{1}_N}{v}\right)\Delta p = \kappa_{(j,\ell)}^{\mathrm{T}}\Delta p,\end{aligned} \tag{20}$$

where in the second equality above, we utilize the fact that $v_j/v_\ell \approx 1, \forall \ell, j \in \mathcal{N}$. Notice that in this case, the power injection sensitivities are independent of the operating point of the network, and they boil down to the current injection sensitivities (see (6)), as expected intuitively.

C. Sensitivities of Line Flows to Line Outages

In this section, we derive a closed-form expression for the sensitivities of line power flows due to line outages. We present the case where there is a single line outage; the case with multiple outages can be derived as a straightforward extension. To this end, consider there is an outage in the (m, n) line. The post-outage conductance matrix is given by

$$\overline{G} := G - g_{mn} e_{mn} e_{mn}^{\mathrm{T}}. \tag{21}$$

Consider the power flow on the (j, ℓ) line as before. The current-injection sensitivity vector, $\overline{\kappa}_{(j,\ell)} \in \mathbb{R}^N$, is

$$\begin{aligned}
\overline{\kappa}_{(j,\ell)}^{\mathrm{T}} &= (g_{j\ell} e_{j\ell}^{\mathrm{T}} + g_j e_j^{\mathrm{T}}) \overline{G}^{-1} \\
&= (g_{j\ell} e_{j\ell}^{\mathrm{T}} + g_j e_j^{\mathrm{T}}) \left(G^{-1} + \frac{g_{mn} G^{-1} e_{mn} e_{mn}^{\mathrm{T}} G^{-1}}{1 - g_{mn} e_{mn}^{\mathrm{T}} G^{-1} e_{mn}} \right) \\
&= \kappa_{(j,\ell)}^{\mathrm{T}} + (g_{j\ell} e_{j\ell}^{\mathrm{T}} + g_j e_j^{\mathrm{T}}) \left(\frac{g_{mn} G^{-1} e_{mn} e_{mn}^{\mathrm{T}} G^{-1}}{1 - g_{mn} e_{mn}^{\mathrm{T}} G^{-1} e_{mn}} \right) \\
&=: \kappa_{(j,\ell)}^{\mathrm{T}} + \Delta \kappa_{(j,\ell)}^{\mathrm{T}},
\end{aligned} \tag{22}$$

where in the second line above, we have utilized the Sherman-Morrison-Woodbury identity [13]; and $\kappa_{(j,\ell)}$ is the pre-outage current-injection sensitivity vector given by (7).

In addition to the change in $\kappa_{(j,\ell)}$ described above, the voltages across the network would also vary as a result of the loss of the (m, n) line. We quantify this next. Beginning with (3), we see that for fixed power injections, we can write

$$\begin{aligned}
p &= \mathrm{diag}\,(v + \Delta v)\, \overline{G}(v + \Delta v) \\
&\approx \mathrm{diag}(v) G v - g_{mn} \mathrm{diag}(v) e_{mn} e_{mn}^{\mathrm{T}} v \\
&\quad + \left(J - g_{mn} \mathrm{diag}(v) e_{mn} e_{mn}^{\mathrm{T}} - \mathrm{diag}(g_{mn} e_{mn} e_{mn}^{\mathrm{T}} v) \right) \Delta v \\
&=: \mathrm{diag}(v) G v - g_{mn} \mathrm{diag}(v) e_{mn} e_{mn}^{\mathrm{T}} v + \overline{J} \Delta v,
\end{aligned} \tag{23}$$

where we have defined

$$\overline{J} := J - g_{mn} \mathrm{diag}(v) e_{mn} e_{mn}^{\mathrm{T}} - \mathrm{diag}(g_{mn} e_{mn} e_{mn}^{\mathrm{T}} v). \tag{24}$$

In the third line of (23), J is the power-flow Jacobian from (12), and we have neglected second-order terms. Since we have $p = \mathrm{diag}(v) G v$, we can isolate Δv from (23) to get

$$\Delta v = \overline{J}^{-1} g_{mn} \mathrm{diag}(v) e_{mn} e_{mn}^{\mathrm{T}} v. \tag{25}$$

We next quantify the sensitivities of line power flows. The power flow along the (j, ℓ) line, $p_{(j,\ell)}$, is given by (14). Assuming the power injections are unchanged, i.e., $\Delta p = \mathbb{0}_N$, we see that the first-order sensitivity of the line power flow to outages is given by

$$\begin{aligned}
p_{(j,\ell)} + \Delta p_{(j,\ell)} &\approx e_j^{\mathrm{T}} v \overline{\kappa}_{(j,\ell)}^{\mathrm{T}}\, \mathrm{diag}\left(\frac{\mathbb{1}_N}{v} \right) p \\
&+ e_j^{\mathrm{T}} \Delta v \overline{\kappa}_{(j,\ell)}^{\mathrm{T}}\, \mathrm{diag}\left(\frac{\mathbb{1}_N}{v} \right) p - e_j^{\mathrm{T}} v \overline{\kappa}_{(j,\ell)}^{\mathrm{T}}\, \mathrm{diag}\left(\frac{\Delta v}{v^2} \right) p.
\end{aligned} \tag{26}$$

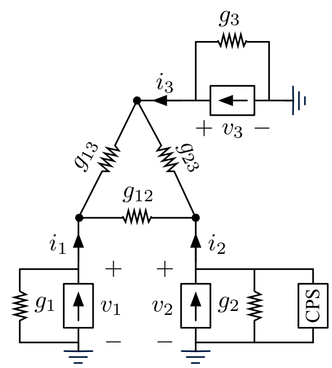

Fig. 1: Network topology for 3-node system.

Substituting $\overline{\kappa}_{(j,\ell)}$ from (22) into the first term of (26), we get

$$\begin{aligned}
& e_j^{\mathrm{T}} v \overline{\kappa}_{(j,\ell)}^{\mathrm{T}}\, \mathrm{diag}\left(\frac{\mathbb{1}_N}{v} \right) p \\
&= e_j^{\mathrm{T}} v \kappa_{(j,\ell)}^{\mathrm{T}}\, \mathrm{diag}\left(\frac{\mathbb{1}_N}{v} \right) p + e_j^{\mathrm{T}} v \Delta \kappa_{(j,\ell)}^{\mathrm{T}}\, \mathrm{diag}\left(\frac{\mathbb{1}_N}{v} \right) p \\
&= p_{(j,\ell)} + e_j^{\mathrm{T}} v \Delta \kappa_{(j,\ell)}^{\mathrm{T}}\, \mathrm{diag}\left(\frac{\mathbb{1}_N}{v} \right) p.
\end{aligned} \tag{27}$$

After substituting (27) and (25) in (26), we then rearrange quantities in the second and third terms in (26) in the same manner as (16) and (17), respectively. We can then express the variation in the power-flow on the (j, ℓ) line, $\Delta p_{(j,\ell)}$, as

$$\begin{aligned}
\Delta p_{(j,\ell)} &= e_j^{\mathrm{T}} v \Delta \kappa_{(j,\ell)}^{\mathrm{T}}\, \mathrm{diag}\left(\frac{\mathbb{1}_N}{v} \right) p \\
&+ \left(\overline{\kappa}_{(j,\ell)}^{\mathrm{T}} \mathrm{diag}\left(\frac{\mathbb{1}_N}{v} \right) p e_j^{\mathrm{T}} - e_j^{\mathrm{T}} v \overline{\kappa}_{(j,\ell)}^{\mathrm{T}}\, \mathrm{diag}\left(\frac{p}{v^2} \right) \right) \\
&\cdot \left(\overline{J}^{-1} g_{mn} \mathrm{diag}(v) e_{mn} e_{mn}^{\mathrm{T}} v \right).
\end{aligned} \tag{28}$$

IV. Case Studies

In this section, we demonstrate the accuracy of the sensitivity expressions of line flows to bus injections and line outages in (19) and (28), respectively, with numerical simulations.

A. Sensitivities of Line Flows to Bus Injections

Figure 1 illustrates the three-node network utilized for this case study. Shunt elements at nodes 1 and 3 are parallel combinations of resistors and constant current sources. The shunt element at node 2 also includes a constant power source (CPS). For simplicity, all conductances (line conductances, g_{ij}, and shunt conductances, g_j) are set to $1\,\Omega^{-1}$. Nominal values of the current sources connected at nodes 1, 2, 3 are 1.0 A, 2.0 A, and 5.0 A, respectively. The power source at node 2 injects a constant value of 2.0 W. With these nominal values, the network voltages are given by $v_1 = 2.39$ V, $v_2 = 2.64$ V, and $v_3 = 3.53$ V. We focus on the power flows on the three lines, $p_{(1,2)}$, $p_{(2,3)}$, and $p_{(3,1)}$ as the power injected by the constant power source is varied. The percentage error between the power flows computed by solving the circuit equations, and those obtained by applying the expression in (19) is computed

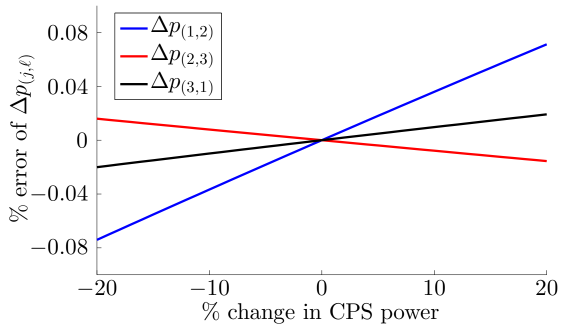

Fig. 2: Error in line flow sensitivities computed from (19) with respect to results from a numerical circuit simulation as the power injected by the CPS is varied from its nominal value of 2 W.

as the power injected by the constant power source is varied between $\pm 20\%$ from its nominal value of 2 W. Results are plotted in Fig. 2 for all three lines; and they establish the accuracy of the approach. In particular, for this case study we observe errors of less than 1% for a $\pm 20\%$ variation in the power injection.

B. Sensitivities of Line Flows to Line Outages

Fig. 3 illustrates the six-node network utilized for this case. The shunt element at nodes 1 and 4 is a constant power load (CPL), while nodes 2 and 6 are CPSs. The shunt element at node 2 also includes a resistor. The conductances of the network are the following: $g_{12} = g_{23} = g_{34} = g_{45} = g_{56} = g_{16} = 1\,\Omega^{-1}$, $g_{13} = g_{15} = g_{35} = 0.5\,\Omega^{-1}$, and $g_2 = 0.001\,\Omega^{-1}$. The nominal values of constant power sources (or loads) at nodes 1, 2, 4, 6 are (negative sign indicates CPL) $-1.0\,\mathrm{kW}$, $2.5\,\mathrm{kW}$, $-2.0\,\mathrm{kW}$, and $1.0\,\mathrm{kW}$, respectively. In this network, single-line outage events are simulated for all cases that do not island the network. We obtain the percentage error between the power flows from the circuit equations and from the sensitivity expression in (28). The average percentage error of the line flow sensitivities for each case is plotted in Fig. 4; the errors are noted to be less than 0.025% for all cases.

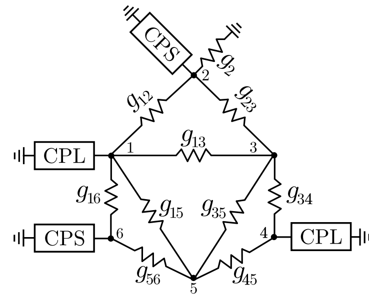

Fig. 3: Network topology for 6-node system.

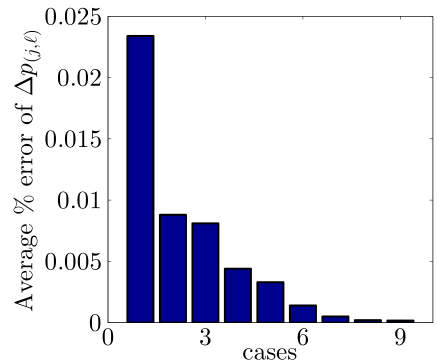

Fig. 4: Average error in line flow sensitivities of single-line outages computed from (28) with respect to those recovered from a numerical simulation for all cases that do not island the network.

V. Concluding Remarks

In this paper, we derived analytical closed-form expressions for the first-order sensitivity of power flows in DC networks to variations in nodal power injections and line outages. The result leveraged current injection sensitivities and a perturbative analysis of matrix-vector representations of power balance expressions. Numerical case studies demonstrated the accuracy of the approach. Future work could attempt to factor higher-order sensitivities to improve accuracy, as well as extend the analytical approach to acknowledge nodal dynamics.

Acknowledgments

This work was supported in part by a Minnesota's Discovery, Research and Innovation Economy (MnDRIVE) grant.

References

[1] T. Gruzs and J. Hall, "Ac, dc or hybrid power solutions for today's telecommunications facilities," in *Telecommunications Energy Conference, 2000. INTELEC. Twenty-second International*, pp. 361–368, 2000.

[2] J. Ciezki and R. Ashton, "Selection and stability issues associated with a navy shipboard dc zonal electric distribution system," *Power Delivery, IEEE Transactions on*, vol. 15, pp. 665–669, Apr 2000.

[3] J. J. Justo, F. Mwasilu, J. Lee, and J.-W. Jung, "Ac-microgrids versus dc-microgrids with distributed energy resources: A review," *Renewable and Sustainable Energy Reviews*, vol. 24, no. 0, pp. 387 – 405, 2013.

[4] J. W. Simpson-Porco, F. Dörfler, and F. Bullo, "On resistive networks of constant power devices," *IEEE Transactions on Systems & Circuits II: Express Briefs*, 2014. To appear.

[5] S. Sanchez, R. Ortega, G. Bergna, M. Molinas, and R. Grino, "Conditions for existence of equilibrium points of systems with constant power loads," in *IEEE 52nd Annual Conference on Decision and Control*, pp. 3641–3646, December 2013.

[6] P. Sauer and M. A. Pai, *Power System Dynamics and Stability*. Upper Saddle River, NJ: Prentice Hall, 1998.

[7] P. Kundur, *Power System Stability and Control*. New York, NY: McGraw-Hill, Inc., 1993.

[8] U.S.-Canada Power System Outage Task Force, "Final report on the august 14th blackout in the united states and canada: causes and recommendations," Apr. 2004.

[9] P. W. Sauer, "On the formulation of power distribution factors for linear load flow methods," *IEEE Transactions on Power App. Syst.*, vol. PAS-100, pp. 764–779, feb 1981.

[10] A. Wood and B. Wollenberg, *Power Generation, Operation and Control*. New York: Wiley, 1996.

[11] P. W. Sauer, K. E. Reinhard, and T. J. Overbye, "Extended factors for linear contingency analysis," in *Proceedings of the 34th Annual Hawaii International Conference on System Sciences*, pp. 697–703, Jan 2001.

[12] T. Güler, G. Gross, and M. Liu, "Generalized line outage distribution factors," *IEEE Transactions on Power Systems*, vol. 22, no. 2, pp. 879–881, 2007.

[13] R. A. Horn and C. R. Johnson, *Matrix Analysis*. New York, NY, USA: Cambridge University Press, 2nd ed., 2012.