Unified Asymptotic Analysis of Linearly Modulated Signals in Fading, Non–Gaussian Noise, and Interference

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Abstract

In this paper, we present a unified asymptotic symbol error rate (SER) analysis of linearly modulated signals impaired by fading and (possibly) non–Gaussian noise, which in our definition also includes interference. The derived asymptotic closed–form results are valid for a large class of fading and noise processes. Our analysis also encompasses diversity reception with equal gain and selection combining and is extended to binary orthogonal modulation. We show that for high signal–to–noise ratios (SNRs) the SER of linearly modulated signals depends on the *Mellin* transform of the probability density function (pdf) of the noise. Since the Mellin transform can be readily obtained for all commonly encountered noise pdfs, the provided SER expressions are easy and fast to evaluate. Furthermore, we show that the diversity gain only depends on the fading statistic and the number of diversity branches, whereas the combining gain depends on the modulation format, the type of fading, the number of diversity branches, and the type of noise. An exception are systems with a diversity gain of one, since their combining gain and asymptotic SER are independent of the type of noise. However, in general, in a log–log scale for high SNR the SER curves for different types of noise are parallel but not identical and their relative shift depends on the Mellin transforms of the noise pdfs.

1 Introduction

The performance of digital communication systems impaired by fading and noise has been extensively studied in the literature, cf. e.g. [1] and references therein. Since analytical expressions for the symbol error rate (SER) are often quite involved, simple yet accurate approximations are desirable for system design [2]. For the asymptotic case of high signal–to–noise ratio (SNR) simple approximations for the SER giving insight into the influence of channel and modulation parameters are available for various modulation schemes, types of fading, and diversity combining techniques, cf. e.g. [3, 4, 5, 6, 7].

All existing asymptotic SER results were obtained for impairment by additive white Gaussian noise (AWGN). However, while AWGN may often be the dominant noise source, there are many practical applications where non–Gaussian noise[1] impairs the received signal. Examples include co–channel and adjacent channel interference in mobile cellular systems [2, 8], impulsive noise in wireless and powerline communications [9], and ultra–wideband (UWB) interference in wireless systems [10]. Although analytical expressions for the SER are available for some types of non–Gaussian noise and interference, a general asymptotic result giving insight into how system performance is affected by the type of noise (in addition to the type of fading) is missing in the literature.

In this paper, we derive simple and elegant asymptotically tight expressions for the SER of linear modulation schemes impaired by fading and (possibly) non–Gaussian noise. Thereby, we assume that the receiver does not know which type of noise is present and applies the detection rule which is optimum for Gaussian noise. The main restriction that we impose on the noise is that the *Mellin transform* $M_z(s)$ [11] of its probability density function (pdf) exists for $s \geq 1$. Most practically relevant types of noise meet this condition.

We also extend our asymptotic SER results to binary orthogonal modulation (BOM), equal gain combining (EGC), and selection combining (SC). Furthermore, we show that for high SNR the SER depends on the Mellin transform $M_z(s)$ of the pdf of the noise process, where the type of fading and the diversity gain of the channel determine the value of the relevant s. Interestingly, we find that the diversity gain of the communication system only depends on the type of fading and the number

[1] To simplify our notation, in this paper, "noise" refers to any additive impairment of the received signal, i.e., our definition of noise also includes what is commonly referred to as "interference".

of diversity branches, whereas the combining gain[2] is also affected by the type of noise. Therefore, in a log–log scale for high SNR the SER curves for different types of noise are parallel. For the special case of a system with a diversity gain of one for high SNR the SER becomes independent of the type of noise.

The remainder of this paper is organized as follows. In Section 2, the considered signal and noise models are introduced. In Section 3, the asymptotic SERs of linear modulation formats and BOM are derived. These results are extended to systems with EGC and SC in Section 4. The derived analytical results are verified by simulations for some representative and relevant special cases in Section 5, and conclusions are drawn in Section 6.

2 Preliminaries

In this section, we present the considered signal, channel, and noise models. However, first we introduce some definitions and notations.

2.1 Some Definitions and Notations

Notation: In this paper, $\Re\{\cdot\}$, $\mathcal{E}\{\cdot\}$, and $\Pr\{A\}$, denote the real part of a complex number, statistical expectation, and the probability of event A, respectively. Furthermore, $I_0(x) \triangleq \frac{1}{2\pi}\int_{-\pi}^{\pi}\exp(x\sin\theta)\,\mathrm{d}\theta$ and $\Gamma(x) \triangleq \int_0^\infty e^{-t}t^{x-1}\,\mathrm{d}t$ are the zeroth order modified Bessel function of the first kind and the Gamma function, respectively. Finally, $\Phi(s) = \mathcal{L}\{p(x)\} \triangleq \int_{-\infty}^{\infty} p(x)e^{-sx}\,\mathrm{d}x$ denotes the Laplace transform of $p(x)$ and a function $f(x)$ is $o(x)$ if $\lim_{x\to\infty} f(x)/x = 0$.

Mellin transform: The Mellin transform $M(s) = \mathcal{M}\{p(x)\} \triangleq \int_0^\infty p(x)x^{s-1}\,\mathrm{d}x$ of a function $p(x)$ will play an important role in this paper. A detailed discussion of the Mellin transform and its properties can be found in Appendix A.

Diversity and combining gain: It is well–known that for transmission over flat fading channels impaired by *Gaussian* noise the SER at high SNR can be approximated by [2, 4]

$$P_E \approx (G_c\,\bar{\gamma})^{-G_d} \tag{1}$$

[2]The combining gain is also sometimes referred to as "coding gain" in the literature, e.g. [4].

where $\bar{\gamma}$ denotes the average SNR, and G_c and G_d are referred to as the *combining gain* and the *diversity gain*, respectively. In this paper, we will show that Eq. (1) is also valid for general non–Gaussian noise.

2.2 Signal Model

For clarity of presentation, we restrict our attention for the moment to the single–receive–branch case. The extension to diversity reception is provided in Section 4. Assuming a frequency–nonselective channel and perfect phase and timing synchronization for the desired signal, the received signal in complex baseband representation can be modeled as

$$r_c = a\, x_c + z_c \tag{2}$$

where a, x_c, and z_c denote the real–valued fading gain, the complex transmitted symbol, and the complex noise, respectively. We assume that a, x_c, and z_c are mutually independent random variables (RVs). The results derived in this paper are applicable to all fading processes whose amplitude pdf $p_a(a)$ can be expanded into a power series around $a = 0$ for high SNR, cf. Section 3. In particular, we will consider Rayleigh, Ricean, Nakagami–m, Nakagami–q, and Weibull fading and the corresponding pdfs $p_a(a)$ are given in Table 1. We emphasize that in general the noise z_c may include both channel noise and interference. Unless stated otherwise[3], in our analysis, we assume that the SER can be obtained by only considering

$$r \triangleq \Re\{r_c\} = a\, x + z \tag{3}$$

where $x \triangleq \Re\{x_c\}$ and $z \triangleq \Re\{z_c\}$. The validity of this assumption is obvious for one–dimensional modulation schemes such as binary phase–shift keying (BPSK) and M–ary pulse amplitude modulation (M–PAM). The same is true for M–ary quadrature amplitude modulation (M–QAM) if the real and imaginary parts of z_c are independent, identically distributed (i.i.d.) RVs [2]. For general M–PSK the above assumption always involves an approximation.

For convenience, we adopt the normalization $\sigma_z^2 \triangleq \mathcal{E}\{z^2\} = 1$, $\bar{\gamma} \triangleq \mathcal{E}\{a^2\}$, and $\sigma_x^2 \triangleq \mathcal{E}\{x^2\} = 2$, i.e., $\bar{\gamma}$ is the average SNR per symbol.

[3] In Section 2.4, for BOM it is necessary to also consider the imaginary part of r_c.

2.3 Admissible Types of Noise and Examples

For the presented asymptotic performance analysis method to be applicable, the RVs z_c and z have to fulfill the following assumptions.

AS1) The pdf $p_z(z)$ of z is an even function, i.e., $p_z(z) = p_z(-z)$.

AS2) The *fundamental strip* of the Mellin transform $M_z(s) \triangleq \mathcal{M}\{p_z(z)\}$ of $p_z(z)$ includes the interval $[1, \infty)$, i.e., $M_z(s)$ exists for $1 \leq \Re\{s\} < \infty$.

AS3) For two–dimensional linear modulation schemes and for BOM we assume that z_c has a rotationally symmetric pdf $p_{z_c}(\cdot)$, i.e., z_c and $e^{j\phi_z} z_c$ have the same pdf for all real ϕ_z.

We note that AS1) is mainly made for convenience as it simplifies our exposition and holds for most types of noise of practical interest. Similar results as in Sections 3 and 4 could also be derived for non–even $p_z(z)$. AS2) is necessary and holds for most practically relevant types of noise. We note, however, that AS2) does not hold for alpha–stable processes with $\alpha < 2$ which have been occasionally used in the past to model impulsive noise, cf. e.g. [12]. For alpha–stable processes, $M_z(s)$ does not exist for $s > \alpha + 1$. AS3) is not necessary for one–dimensional linear modulation schemes but considerably simplifies the asymptotic analyses of two–dimensional modulation formats and BOM, respectively.

Gaussian noise obviously fulfills AS1)–AS3). In the following, we will briefly discuss three relevant non–Gaussian types of noise which also fulfill at least AS1) and AS2). The pdfs $p_z(z)$, Laplace transforms $\Phi_z(s) \triangleq \mathcal{L}\{p_z(z)\}$, and Mellin transforms $M_z(s)$ for these noises as well as those for Gaussian noise and generalized Gaussian noise are summarized in Table 2.

E1) Gaussian mixture noise: Gaussian mixture noise is often used to model the combined effect of Gaussian background noise and man–made or impulsive noise, cf. e.g. [9, 13, 14]. In this case, the pdf of z_c is given by

$$p_{z_c}(z_c) = \sum_{k=1}^{I} \frac{c_k}{2\pi\sigma_{z_k}^2} \exp\left(-\frac{|z_c|^2}{2\sigma_{z_k}^2}\right) \tag{4}$$

where $c_k > 0$ and $\sigma_{z_k}^2 > 0$ are parameters. Two popular special cases of Gaussian mixture noise are Middleton's Class–A noise [13] and ϵ–mixture noise. For ϵ–mixture noise $I = 2$, $c_1 = 1 - \epsilon$, $c_2 = \epsilon$, $\sigma_{z_1}^2 = \sigma_g^2$, and $\sigma_{z_2}^2 = \kappa\sigma_g^2$, where ϵ is the fraction of time when the impulsive noise is present, κ is the ratio of the variances of the Gaussian background noise and the impulsive noise, and $\sigma_g^2 = 1/(1-\epsilon+\kappa\epsilon)$. It is easy to verify that AS1)–AS3) are valid for Gaussian mixture noise.

E2) BPSK interference with fixed channel phase: The complex and real interference (noise) from I independent, symbol synchronous[4] BPSK signals $i_k \in \{\pm 1\}$, $1 \leq k \leq I$, can be modeled as

$$z_c = \sum_{k=1}^{I} d_{c,k}\, i_k \qquad \text{and} \qquad z = \sum_{k=1}^{I} d_k\, i_k \tag{5}$$

respectively, where $d_{c,k} \triangleq |d_{c,k}|\, e^{j\varphi_{d_{c,k}}}$ and $d_k \triangleq |d_{c,k}| \cos(\varphi_{d_{c,k}})$ denote the complex and the real gain of the kth interference channel, respectively. The interference channel phases $\varphi_{d_{c,k}}$ are assumed to be constant. We note that BPSK interference with fixed channel phase only fulfills A1) and A2), i.e., the validity of the presented asymptotic analysis is restricted to one–dimensional linear modulation schemes in this case.

E3) M–PSK interference with random channel phase: In this case, z_c and z are also given by Eq. (5) but the phases $\varphi_{d_{c,k}}$, $1 \leq k \leq I$, are mutually independent RVs uniformly distributed in the interval $(-\pi,\, \pi]$ and $i_k \in \{e^{j2\pi m/M} | m \in \{0,\, 1,\, \ldots,\, M-1\}\}$. The randomness of the phases $\varphi_{d_{c,k}}$ may be due to e.g. the lack of phase synchronization between the interferers and the desired signal. Because of the uniformly distributed phases, AS3) holds in addition to AS1) and AS2).

2.4 Mellin Transform of Composite Noise

In general, the noise z may be the sum or the product of different RVs z_k, $1 \leq k \leq I$. In this case, the framework developed in this paper is applicable as long as AS1)–AS3) hold and we explain in Appendix A how the Mellin transform $M_z(s)$ of $p_z(z)$ can be obtained from the Mellin transforms of the pdfs of z_k, $1 \leq k \leq I$. To illustrate the application of the results in Appendix A, we briefly discuss two practically relevant examples.

E4) Ricean faded M–PSK interferer: A Ricean faded M–PSK interferer can be modeled as $z = z_1 + z_2$, where z_1 and z_2 represent the direct and the specular (Rayleigh) component, respectively.

[4] We note that even if the BPSK interference signals are not symbol synchronous with the desired signal, for z_c and z a similar model as in Eq. (5) results, cf. e.g. [8], and the mathematical tools developed in this paper are still applicable. However, because of space limitations, we only consider the symbol synchronous case here.

z_1 can be modeled by Example E3) and z_2 is a real Gaussian RV. The Mellin transform $M_z(s)$ can be obtained by applying Eq. (50)[5], where $M_{z_1}(s)$ and $M_{z_2}(s)$ are given in Table 2.

E5) Rayleigh faded multiple BPSK interferers: If multiple synchronous BPSK interference signals i_k, $1 \leq k \leq I$, originate from the same transmitter (e.g. base station), they arrive over the same channel at the receiver (e.g. moblile station or base station of another cell) for the desired signal. Therefore, if the interference channel is Rayleigh faded, this type of interference can be modeled as $z = z_1 z_2$, where z_1 and z_2 represent the fading gain (real Gaussian RV) and the interference signal (modeled as in Example E2)), respectively.[6] In this case, $M_z(s)$ can be obtained from Eq. (46), where $M_{z_1}(s)$ and $M_{z_2}(s)$ are again given in Table 2. This interference model applies for example to synchronous code–division multiple access (CDMA) systems after despreading where the coefficients $d_{c,k}$ in Eq. (5) denote the correlation of the desired user's signature sequence with the signature sequences of the users in an neighboring cell [2].

For more complicated types of noise Eqs. (46) and (50) may have to be applied repeatedly. Alternatively, in cases where a closed–form expression for $M_z(s)$ cannot be found or if only measurements of z are available, the Mellin transform $M_z(s)$ may also be estimated using Monte Carlo integration of Eq. (42). Exploiting that $p_z(z)$ is an even function, an estimate of $M_z(s)$ is given by

$$\hat{M}_z(s) = \frac{1}{N_z} \sum_{k=1}^{N_z} |z[k]|^{s-1} \tag{6}$$

where $z[k]$, $1 \leq k \leq N_z$, are realizations of the RV z. For a sufficiently large number N_z of samples the estimate $\hat{M}_z(s)$ will approach $M_z(s)$. Of course, the validity of AS2) has to be verified. However, this can often be accomplished without knowing $p_z(z)$ or $M_z(s)$ explicitly.

3 Single–Branch Reception

In this section, we develop exact and asymptotic expressions for the SER of linear modulation schemes such as M–PAM, M–PSK, and M–QAM with a single diversity branch. In addition, we

[5]We note that z_1 and z_2 are statistically independent although they involve the same M–PSK interference signal.

[6]It is interesting to note that a similar interference model as in E5) also holds for an asynchronous Rayleigh faded BPSK interferer, cf. [8].

also consider the asymptotic SER of BOM.

3.1 Basic Error Probability Result

The calculation of the SER of linear modulation schemes involves the evaluation of the probability $P_e(d)$ that the received signal $r = ax+z$ is smaller than a certain threshold ay, where $d \triangleq x-y > 0$. Conditioned on a we obtain

$$P_e(d|a) = \Pr\{ax + z < ay|a\} = \int_{-\infty}^{-ad} p_z(z)\,\mathrm{d}z = \frac{1}{2\pi j}\int_{c-j\infty}^{c+j\infty} \Phi_z(s)\,e^{-ads}\,\frac{\mathrm{d}s}{s} \tag{7}$$

where $\Phi_z(s) \triangleq \mathcal{L}\{p_z(z)\}$ and constant c lies in the region of convergence of the integral. Averaging $P_e(d|a)$ with respect to a yields

$$P_e(d) = \mathcal{E}\{P_e(d|a)\} = \frac{1}{2\pi j}\int_{c-j\infty}^{c+j\infty} \Phi_z(s)\,\mathcal{E}\{e^{-ads}\}\,\frac{\mathrm{d}s}{s} = \frac{1}{2\pi j}\int_{c-j\infty}^{c+j\infty} \Phi_z(s)\,\Phi_a(ds)\,\frac{\mathrm{d}s}{s} \tag{8}$$

where $\Phi_a(s) \triangleq \mathcal{L}\{p_a(a)\}$. Since for most practically relevant cases, both $\Phi_a(s)$ (cf. [15, Table I]) and $\Phi_z(s)$ (cf. Table 2) are readily available, Eq. (8) can be efficiently evaluated numerically applying e.g. a Gauss–Chebyshev quadrature rule, cf. e.g. [16]. However, unfortunately this numerical approach does not provide any intuitive insight into the problem. Therefore, using Parseval's theorem we rewrite Eq. (8) as

$$P_e(d) = \int_{0}^{\infty} p_z(u) F_a\left(\frac{u}{d}\right)\,\mathrm{d}u \tag{9}$$

where $F_a(a) \triangleq \mathcal{L}^{-1}\{\Phi_a(s)/s\} = \int_0^a p_a(u)\,\mathrm{d}u$ is the cumulative distribution function (cdf) of a. To further simplify Eq. (9), we assume that for high SNR $\bar{\gamma}$ the pdf $p_a(a)$ can be expanded into a series

$$p_a(a) = \frac{1}{a}\left(\sum_{k=1}^{N} p_k \left(\frac{a^2}{\bar{\gamma}}\right)^{\xi k+\delta} + o\left((a^2/\bar{\gamma})^{\xi N+\delta}\right)\right), \qquad a \geq 0 \tag{10}$$

where ξ and δ are real–valued constants and p_k are real–valued coefficients. In Eq. (10), N is the number of terms considered and $o\left((a^2/\bar{\gamma})^{\xi N+\delta}\right)$ is a remainder term. For Rayleigh, Ricean, Nakagami–q, Nakagami–m, and Weibull fading ξ, δ, and p_k, $k \geq 1$, are given in Table 1. With Eq. (10) the cdf $F_a(a)$ can be expressed as

$$F_a(a) = \frac{1}{2}\sum_{k=1}^{N} \frac{p_k}{\xi k+\delta}\left(\frac{a^2}{\bar{\gamma}}\right)^{\xi k+\delta} + o\left((a^2/\bar{\gamma})^{\xi N+\delta}\right). \tag{11}$$

Applying Eq. (11) in Eq. (9) and assuming that AS2) in Section 2.3 holds, we can express the error probability as

$$P_e(d) = \frac{1}{2} \sum_{k=1}^{N} \frac{p_k\, M_z(2(\xi k + \delta) + 1)}{\xi k + \delta} \, (d^2 \bar{\gamma})^{-(\xi k + \delta)} + o\left(\bar{\gamma}^{-(\xi N + \delta)}\right). \tag{12}$$

For high SNR the remainder term $o\left(\bar{\gamma}^{-(\xi N+\delta)}\right)$ becomes negligible and the first term in Eq. (12) gives a tight approximation of $P_e(d)$. Furthermore, for high enough SNR the series in Eq. (12) converges for $N \to \infty$. In this case, Eq. (12) is equivalent to Eq. (8). The exact value of $\bar{\gamma}$ where the series starts to converge depends on both the type of fading (via p_k) and the type of noise [via $M_z(2(\xi k + \delta) + 1)$].

3.2 Exact SER Expression for M–PAM

Although the main emphasis of this paper is on SER approximations offering insight into the system behavior at high SNR, it is worth noting that for one–dimensional modulation schemes such as BPSK and M–PAM, $P_e(d)$ in Eq. (12) can be used to derive an expression for the exact SER. In particular, using similar steps as for the Gaussian case in [2, Ch. 5] and assuming that the SNR is high enough for Eq. (12) to converge for $N \to \infty$, we obtain

$$P_M^{\mathrm{PAM}} = 2\, \beta_M^{\mathrm{PAM}}\, P_e\left(d_M^{\mathrm{PAM}}\right) = \beta_M^{\mathrm{PAM}} \sum_{k=1}^{\infty} \frac{p_k\, M_z(2(\xi k + \delta) + 1)}{\xi k + \delta} \, ((d_M^{\mathrm{PAM}})^2 \bar{\gamma})^{-(\xi k + \delta)} \tag{13}$$

where β_M^{PAM} and d_M^{PAM} are given in Table 3. The SER of BPSK P_2^{PSK} can be obtained from $P_2^{\mathrm{PSK}} = P_2^{\mathrm{PAM}}$. To verify the result in Eq. (13), we can consider the special case of $M = 2$, Nakagami–m fading, and Gaussian noise. Using ξ, δ, and p_k from Table 1 and $M_z(s)$ from Table 2, it is straightforward to show that under these conditions Eq. (13) can be simplified to [4, Eq. (9)]. We emphasize that the technique in [4] is limited to Gaussian noise, whereas Eq. (13) is valid for any type of noise fulfilling AS1) and AS2) in Section 2.3.

We note that P_M^{PAM} can be evaluated numerically also by combining Eqs. (8) and (13). This numerical approach also succeeds at low SNRs, where Eq. (12) does not converge, but does not reveal the connection between the Mellin transform of $p_z(z)$ and the SER.

3.3 Asymptotic SER of Linear Modulations

For high SNR, we can approximate $P_e(d)$ by the first term of the sum in Eq. (12), i.e.,

$$P_e(d) \approx \frac{1}{2} \frac{p_1 \, M_z(2(\xi+\delta)+1)}{\xi+\delta} \, (d^2 \bar{\gamma})^{-(\xi+\delta)}. \tag{14}$$

Based on this result the asymptotic SER of linear modulation schemes can be approximated as

$$P_M^{\mathrm{X}} \approx \beta_M^{\mathrm{X}} \, \frac{p_1 \, M_z(2(\xi+\delta)+1)}{\xi+\delta} \, ((d_M^{\mathrm{X}})^2 \bar{\gamma})^{-(\xi+\delta)} \tag{15}$$

where X stands for PAM, PSK, and QAM, respectively. The respective values of β_M^{X} and d_M^{X} are summarized in Table 3, cf. also [2]. We note that for M–PSK with $M \geq 4$ and M–QAM AS3) in Section 2.3 is necessary to ensure that for high enough SNR β_M^{X} and d_M^{X} are independent from the pdf of z.

Eq. (15) shows that Eq. (1) not only holds for Gaussian noise but also for the more general class of noises considered in this paper. In particular, comparing Eq. (1) and Eq. (15), the diversity gain is $G_d = \xi + \delta$ and the combining gain is given by

$$G_c = (d_M^{\mathrm{X}})^2 \left(\frac{G_d}{\beta_M^{\mathrm{X}} \, p_1 \, M_z(2G_d+1)} \right)^{1/G_d}. \tag{16}$$

Therefore, for high SNR in a log–log scale the slope of the SER curves ($-G_d$) only depends on the fading statistic but is independent of the noise statistic. On the other hand, the relative shift of the SER curves (G_c) depends on both the fading and the noise statistics. The pdf $p_z(z)$ of the noise z influences the combining gain via its Mellin transform $M_z(s)$ for $s = 2G_d + 1$. Since $G_d = \xi + \delta$ depends on the fading process, the fading statistic also determines in part what effect the type of noise has on G_c and on the asymptotic SER. To further highlight this point, we specialize Eq. (15) in the following.

1) Rayleigh, Ricean, and Nakagami–q fading: In this case, $\xi = 1$ and $\delta = 0$, cf. Table 1. Since $M_z(3) = \sigma_z^2/2 = 1/2$ holds always, we obtain

$$P_M^{\mathrm{X}} \approx \frac{\beta_M^{\mathrm{X}} \, p_1}{2(d_M^{\mathrm{X}})^2 \, \bar{\gamma}} \tag{17}$$

i.e., surprisingly the asymptotic SER for Rayleigh, Ricean, and Nakagami–q fading is *independent* of the noise statistic. If we further specialize Eq. (17) to BPSK and Rayleigh fading, we obtain

$P_2^{\mathrm{PSK}} = 1/(4\bar{\gamma})$, which is a famous result for Gaussian noise [2]. However, our analysis here shows that this result is also valid for a much larger class of noises.

2) Nakagami–m fading: Using $\xi = 1$, $\delta = m-1$, and $p_1 = 2m^m/\Gamma(m)$ in Eq. (15) yields

$$P_M^{\mathrm{X}} \approx \frac{2\,\beta_M^{\mathrm{X}}\,m^{m-1}\,M_z(2m+1)}{\Gamma(m)\,(d_M^{\mathrm{X}})^{2m}\,\bar{\gamma}^m} \tag{18}$$

i.e., the SER depends on $M_z(2m+1)$. Therefore, in a log–log scale non–Gaussian noise will result in a horizontal shift of the SER curve by

$$G_N(m) \triangleq \frac{10}{m}\,\log_{10}\left(\frac{\sqrt{\pi}M_z(2m+1)}{\Gamma(m-1/2)2^{m-1}}\right)\,\mathrm{dB} \tag{19}$$

compared to Gaussian noise. If $G_N(m)$ is negative, the SER caused by the non–Gaussian noise is lower than that caused by Gaussian noise. The opposite is true if $G_N(m)$ is positive. For the special case of ϵ–mixture noise, Eq. (19) simplifies to

$$G_N(m) = 10\,\log_{10}\left(\frac{\sqrt[m]{1-\epsilon+\epsilon\kappa^m}}{1-\epsilon+\epsilon\kappa}\right)\,\mathrm{dB}. \tag{20}$$

For example, for $\epsilon = 0.01$ and $\kappa = 100$, we obtain $G_N(0.5) = -2.24$ dB, $G_N(1) = 0$ dB, $G_N(2) = 7.03$ dB, and $G_N(3) = 10.34$ dB, which clearly shows that given the same noise statistic, different fading statistics may cause significantly different combining gains.

3) Weibull fading: Adopting $\xi = c/2$, $\delta = 0$, and $p_1 = (\Gamma(1+2/c))^{c/2}$, we obtain

$$P_M^{\mathrm{X}} \approx \frac{2\,\beta_M^{\mathrm{X}}\,(\Gamma(1+2/c))^{c/2}\,M_z(c+1)}{c\,(d_M^{\mathrm{X}})^c\,\bar{\gamma}^{c/2}} \tag{21}$$

i.e., the asymptotic SER depends on $M_z(c+1)$. Similar to the Nakagami–m fading case, non–Gaussian noise causes a horizontal shift of the SER curve. A comparison of Eqs. (18) and (21) shows that for Weibull fading with parameter c this shift is simply given by $G_N(c/2)$, i.e., the SER curves of Nakagami–m fading with parameter m and those of Weibull fading with parameter $c = 2m$ are shifted by the same amount if the noise is non–Gaussian instead of Gaussian.

3.4 Asymptotic SER of BOM

Using the definitions $r \triangleq \Re\{r_c\}$ and $\bar{r} \triangleq \Im\{r_c\}$, in BOM, the output of the two correlators is $\{r = \sqrt{2}a + z,\, \bar{r} = \bar{z}\}$ and $\{r = z,\, \bar{r} = \sqrt{2}a + \bar{z}\}$ if bit "1" and "0" are transmitted, respectively, where $z \triangleq \Re\{z_c\}$ and $\bar{z} \triangleq \Im\{z_c\}$ [2]. Assuming coherent detection, the decision variable can be defined as

$$\tilde{r} \triangleq r - \bar{r} = a\,x + \tilde{z} \tag{22}$$

where $x \in \{\pm\sqrt{2}\}$ and $\tilde{z} \triangleq z - \bar{z}$. Comparing Eq. (22) with Eq. (3) it is obvious that the framework developed in Sections 3.2 and 3.3 is also applicable to BOM. For calculation of the Mellin transform $M_{\tilde{z}}(s)$ of $\tilde{z}$ we note that $\tilde{z}$ can be expressed as

$$\tilde{z} = |z_c|[\cos(\Theta_c) - \sin(\Theta_c)] = \sqrt{2}\,|z_c|\cos(\Theta_c + \pi/4) \tag{23}$$

where Θ_c denotes the phase of z_c. Since according to AS3) in Section 2.3 Θ_c is uniformly distributed $z = |z_c|\cos(\Theta_c)$ and $|z_c|\cos(\Theta_c + \pi/4)$ have the same pdf. Therefore, from Eq. (45) we obtain $M_{\tilde{z}}(s) = (\sqrt{2})^{s-1}\,M_z(s)$. Using this result in Eq. (15), we obtain

$$P_2^{\mathrm{OM}} \approx \beta_2^{\mathrm{OM}}\,\frac{p_1\,M_z(2(\xi+\delta)+1)}{\xi+\delta}\,(2\,(d_2^{\mathrm{OM}})^2\bar{\gamma})^{-(\xi+\delta)} \tag{24}$$

where β_2^{OM} and d_2^{OM} are given in Table 3. Since $\beta_2^{\mathrm{OM}} = \beta_2^{\mathrm{PSK}}$ and $d_2^{\mathrm{OM}} = d_2^{\mathrm{PSK}}$, a comparison of Eq. (15) specialized to BPSK and Eq. (24) shows that BOM suffers from an SNR loss of 3 dB compared to BPSK. While this 3 dB loss is a well–known fact for Gaussian noise [2], our analysis here shows that it also holds for a much larger class of noises independent of the fading statistic. For completeness we note that the exact SER of BOM can be obtained by replacing $M_z(s)$ by $M_{\tilde{z}}(s)$ in Eq. (13).

4 Diversity Combining

In this section, we extend the framework introduced in Section 3 to equal gain combining (EGC) and selection combining (SC). We assume that the signal model in Eq. (3) is valid for L diversity branches, i.e.,

$$r_l = a_l\,x + z_l, \qquad 1 \le l \le L \tag{25}$$

where r_l, a_l, and z_l denote the received signal, the fading amplitude, and the noise in the lth branch. Furthermore, we assume $\sigma_z^2 \triangleq \mathcal{E}\{z_l^2\} = 1$, $1 \le l \le L$, and the SNR of the lth branch is $\bar{\gamma}_l \triangleq \mathcal{E}\{a_l^2\}$, i.e., different branches may have different SNRs. For convenience, we assume that the fading gains in different branches are statistically independent and follow the same distribution (e.g. all branches are Nakagami–m distributed with the same m). The latter restriction is only made to arrive at simple and insightful results and the extension to the case where different branches follow different distributions is straightforward. We also assume that the noise RVs z_l, $1 \leq l \leq L$, have the same pdf $p_z(z)$ and fulfill AS1)–AS3) in Section 2.3.

4.1 Equal Gain Combining (EGC)

In coherent EGC the complex received signals of all branches are co–phased and combined. The resulting decision variable is given by

$$\tilde{r} = \sum_{l=1}^{L} r_l = \tilde{a}x + \tilde{z} \tag{26}$$

where $\tilde{a} \triangleq \sum_{l=1}^{L} a_l$ and $\tilde{z} \triangleq \sum_{l=1}^{L} z_l$.

For the framework developed in Section 3 to be applicable to EGC, we require the series expansion of the pdf $p_{\tilde{a}}(a)$ of $\tilde{a}$ and the Mellin transform $M_{\tilde{z}}(s)$ of the pdf of $\tilde{z}$. For general s and mutually dependent z_l, $1 \leq l \leq L$, the calculation of $M_{\tilde{z}}(s)$ may be quite involved. However, for the most important special case where the z_l, $1 \leq l \leq L$, are mutually statistically independent and s is an integer, $M_{\tilde{z}}(s)$ can be easily obtained by applying Eq. (50).

For the series expansion of $p_{\tilde{a}}(a)$ we first note that the Laplace transform of $p_{\tilde{a}}(a)$ can be expressed as [15]

$$\Phi_{\tilde{a}}(s) \triangleq \mathcal{L}\{p_{\tilde{a}}(a)\} = \prod_{l=1}^{L} \Phi_{a_l}(s) \tag{27}$$

where $\Phi_{a_l}(s) \triangleq \mathcal{L}\{p_{a_l}(a)\}$ and $p_{a_l}(a)$ denotes the pdf of a_l. Considering Eq. (10) $\Phi_{a_l}(s)$ can be expressed as

$$\Phi_{a_l}(s) = \sum_{k=1}^{N} p_k \, \Gamma(2(\xi k + \delta)) \, (s^2 \bar{\gamma}_l)^{-(\xi k + \delta)} + o\left((s^2 \bar{\gamma}_l)^{-(\xi N + \delta)}\right). \tag{28}$$

By combining Eqs. (27) and (28) we can obtain a series expansion for $\Phi_{\tilde{a}}(s)$ which then can be used to obtain the desired series expansion for $p_{\tilde{a}}(a)$. Since the general expression for this expansion is quite involved and we are mainly interested in asymptotic results, we restrict our attention to the first term of the expansion

$$p_{\tilde{a}}(a) = \frac{1}{a}\left[p_1^L \frac{[\Gamma(2(\xi+\delta))]^L}{\Gamma(2L(\xi+\delta))}\prod_{l=1}^{L}\left(\frac{a^2}{\bar{\gamma}_l}\right)^{\xi+\delta} + o\left(\prod_{l=1}^{L}\left(a^2/\bar{\gamma}_l\right)^{\xi+\delta}\right)\right]. \tag{29}$$

Using Eq. (29) for EGC the basic error probability $P_e(d)$ defined in Section 3.1 can be expressed as

$$P_e(d) = \frac{p_1^L\,[\Gamma(2(\xi+\delta))]^L\, M_{\tilde{z}}(2L(\xi+\delta)+1)}{2L\,(\xi+\delta)\,\Gamma(2L(\xi+\delta))}\prod_{l=1}^{L}(d^2\bar{\gamma}_l)^{-(\xi+\delta)} + o\left(\prod_{l=1}^{L}(\bar{\gamma}_l)^{-(\xi+\delta)}\right) \tag{30}$$

where $d = x - y > 0$ as before. Thus, the asymptotic SER of linear modulation schemes with EGC can be approximated as

$$P_M^{\mathrm{X}} \approx \beta_M^{\mathrm{X}}\, \frac{p_1^L\,[\Gamma(2(\xi+\delta))]^L\, M_{\tilde{z}}(2L(\xi+\delta)+1)}{L\,(\xi+\delta)\,\Gamma(2L(\xi+\delta))}\prod_{l=1}^{L}\left((d_M^{\mathrm{X}})^2\bar{\gamma}_l\right)^{-(\xi+\delta)}. \tag{31}$$

Specializing this result for BPSK to Ricean fading and Gaussian noise leads to $P_2^{\mathrm{PSK}} \approx (1+K)^L e^{-KL} L^L/[2^{L-1}L!\prod_{l=1}^{L}\bar{\gamma}_l]$, which is in perfect agreement with [3, Eq. (5)], [1, Eq. (9.38)].

Assuming for simplicity equal branch SNRs $\bar{\gamma}_l = \bar{\gamma}$, $1 \le l \le L$, the diversity gain follows as $G_d = L(\xi+\delta)$, whereas the combining gain is given by

$$G_c = (d_M^{\mathrm{X}})^2\left(\frac{G_d\,\Gamma(2G_d)}{\beta_M^{\mathrm{X}}\,p_1^L\,[\Gamma(2G_d/L)]^L\,M_{\tilde{z}}(2G_d+1)}\right)^{1/G_d}. \tag{32}$$

From Eqs. (31) and (32) we observe that while the slope of the SER curves is not influenced by the type of noise, the SER curve for non–Gaussian noise experiences a horizontal shift compared to the SER curve for Gaussian noise. This horizontal shift corresponds to a difference in the combining gain and depends on $M_{\tilde{z}}(2G_d+1)$. For Rayleigh, Ricean, and Nakagami–q fading $G_d = L$ and the Mellin transform $M_{\tilde{z}}(2L+1)$ (and therefore also the SER and G_c) depends on the type of noise if $L > 1$.

4.2 Selection Combining (SC)

In SC, only the path with the largest fading amplitude is considered for detection. Therefore, the decision variable can be modeled as

$$\check{r} \triangleq \check{a}\,x + z \tag{33}$$

where $\check{a} \triangleq \max\{a_1, a_2, \ldots, a_L\}$ and z has the same pdf $p_z(z)$ as z_l, $1 \leq l \leq L$. Since the cdf of $\check{a}$ is given by $F_{\check{a}}(a) = \prod_{l=1}^{L} F_{a_l}(a)$ [1], where $F_{a_l}(a)$ is the cdf of a_l, we can obtain the expansion

$$F_{\check{a}}(a) = \frac{p_1^L}{2^L(\xi+\delta)^L} \prod_{l=1}^{L} \left(\frac{a^2}{\bar{\gamma}_l}\right)^{\xi+\delta} + o\left(\prod_{l=1}^{L} \left(a^2/\bar{\gamma}_l\right)^{\xi+\delta}\right) \tag{34}$$

cf. Eq. (11). Therefore, the basic error probability $P_e(d)$ defined in Section 3.1 is now given by

$$P_e(d) = \frac{p_1^L\, M_z(2L(\xi+\delta)+1)}{2^L(\xi+\delta)^L} \prod_{l=1}^{L} \left(d^2\,\bar{\gamma}_l\right)^{-(\xi+\delta)} + o\left(\prod_{l=1}^{L} \left(\bar{\gamma}_l\right)^{-(\xi+\delta)}\right). \tag{35}$$

Exploiting Eq. (35), we can express the asymptotic SER of linear modulations as

$$P_M^{\mathrm{X}} \approx \beta_M^{\mathrm{X}}\, \frac{p_1^L\, M_z(2L(\xi+\delta)+1)}{2^{L-1}(\xi+\delta)^L} \prod_{l=1}^{L} \left((d_M^{\mathrm{X}})^2\,\bar{\gamma}_l\right)^{-(\xi+\delta)}. \tag{36}$$

For BPSK transmission over a Rayleigh faded channel with Gaussian noise Eq. (36) simplifies to $P_2^{\mathrm{PSK}} = (2L)!/\,[2^{2L-1}L!\,\prod_{l=1}^{L}\bar{\gamma}_l]$, which can be shown to be identical to [1, Eq. (9.268)] for $L=2$ and high SNR.[7]

If we assume again equal branch SNRs $\bar{\gamma}_l = \bar{\gamma}$, $1 \leq l \leq L$, from Eq. (36) we obtain a diversity gain of $G_d = L(\xi+\delta)$ and a combining gain of

$$G_c = (d_M^{\mathrm{X}})^2 \left(\frac{2^{L-1}G_d^L}{\beta_M^{\mathrm{X}}\, p_1^L\, L^L\, M_z(2G_d+1)}\right)^{1/G_d}. \tag{37}$$

Similar to the EGC case for $L > 1$ the SER and the combining gain depend on the type of noise also for Rayleigh, Ricean, and Nakagami–q fading.

4.3 Comparison of EGC and SC

It is interesting to compare the combining gains achievable with EGC and SC. For this purpose, we define the relative gain $G_c^{\mathrm{E/S}}(L)$ as the ratio of Eqs. (32) and (37)

$$G_c^{\mathrm{E/S}}(L) = \left(\frac{L^L\,\Gamma(2G_d)}{(2G_d)^{L-1}[\Gamma(2G_d/L)]^L}\,\frac{M_z(2G_d+1)}{M_{\tilde{z}}(2G_d+1)}\right)^{1/G_d} \tag{38}$$

i.e., for high enough SNRs EGC achieves a gain of $10\log_{10}(G_c^{\mathrm{E/S}})$ dB over SC. The first term on the right hand side of Eq. (38) is only influenced by the type of fading, whereas the second term

[7] More precisely, the variables ρ and g, which are defined in [1], have to be set to $\rho = 0$ and $g = 1$ in [1, Eq. (9.268)] to obtain $P_2^{\mathrm{PSK}} \approx 3/[8\bar{\gamma}_1\bar{\gamma}_2]$ for high SNR.

is affected by both the type of noise and the type of fading. If we assume Rayleigh, Ricean, or Nakagami–q fading, $G_d = L$ and Eq. (38) simplifies to

$$G_c^{\mathrm{E/S}}(L) = \left(\frac{(2L)!\, M_z(2L+1)}{2^L\, M_{\tilde{z}}(2L+1)} \right)^{1/L}. \tag{39}$$

If we furthermore assume Gaussian noise, we obtain $G_c^{\mathrm{E/S}}(L) = [(2L)!/(2^L L^2)]^{1/L} > 1$, i.e., EGC always outperforms SC. For dual diversity and i.i.d. noise RVs z_1 and z_2 Eq. (39) can be simplified to

$$G_c^{\mathrm{E/S}}(2) = \sqrt{\frac{6M_z(5)}{2M_z(5)+3}} \tag{40}$$

where Eq. (50) and $M_z(3) = 1/2$ have been exploited. As an example, we may consider the case of an interference limited system where z_1 and z_2 are due to a Ricean faded M–PSK interferer with Ricean factor K_I and uniformly distributed channel phase[8]. The Mellin transform $M_z(5)$ for this case can be calculated as explained in Example E4) in Section 2.4. The resulting $G_c^{\mathrm{E/S}}(2)$ is

$$G_c^{\mathrm{E/S}}(2) = \sqrt{\frac{6+12K_I+3K_I^2}{4+8K_I+3K_I^2}} \tag{41}$$

which is a monotonic decreasing function in K_I. For example, from Eq. (41) we obtain that the performance gain of EGC compared to SC is 0.88 dB, 0.23 dB, and 0 dB for $K_I = 0$, $K_I = 10$, and $K_I \to \infty$, respectively.

5 Numerical Results and Discussions

In this section, we verify the derived analytical expressions for the asymptotic SER for different practically relevant cases. First, the case of a single diversity branch is considered, then results for diversity combining are presented.

5.1 Single–Branch Reception

Fig. 1 shows the SER of 8–PSK modulation in Nakagami–m fading and ϵ–mixture noise ($\epsilon = 0.25$, $\kappa = 10$). As expected, for high enough SNR the simulation curves (markers) closely approach the

[8]We note that since the two interference processes are i.i.d. and the pdf of Ricean fading with uniformly distributed channel phase is rotationally symmetric, z_1 and z_2 are statistically independent although they involve the same M–PSK interference signal.

asymptotic results obtained from our analysis (solid lines).

In Fig. 2, we show the SER (which is identical to the bit error rate in this case) for BPSK modulation in Nakagami–m fading ($m = 2$) for some types of noise discussed in Sections 2.3 and 2.4. Fig. 2 clearly illustrates that for a given SNR the SER caused by non–Gaussian noise and interference may be considerably lower or higher than that caused by Gaussian noise.

Fig. 3 shows the SER of a narrowband (NB) signal having bandwidth B_s and employing 16–QAM in Nakagami–m fading ($m = 2$) with a multi–band orthogonal frequency division multiplexing (MB–OFDM) and a direct–sequence ultra–wideband (DS–UWB) interferer, respectively. The NB pulse shape is a square–root raised cosine filter with roll–off factor 0.35 and the receiver employs the corresponding matched filter. The MB–OFDM and DS–UWB interferers were generated according to the corresponding IEEE 802.15.3a standard proposals [17, 18]. Since a closed–form calculation of the related Mellin transforms is too involved, we estimated $M_z(s = 5)$ using Eq. (6) and then calculated the asymptotic SER using Eq. (18). This semi–analytical approach is much faster than directly simulating the SER. Interestingly, Fig. 3 shows that while for $B_s = 1$ MHz the MB–OFDM interferer yields a lower SER than the DS–UWB interferer, the opposite is true for $B_s = 20$ MHz.

5.2 Diversity Combining

Fig. 4 shows the SER of BPSK with EGC and SC, respectively, in Ricean fading ($K = 2$). We assume $L = 2$ and identical SNRs for both diversity branches. The BPSK signal is impaired by a Ricean faded M–PSK interferer with Ricean factor K_I and uniformly distributed phase, cf. Sections 2.4, 4.3. The relative performance loss of SC compared to EGC is smaller for $K_I = 10$ than for $K_I = 0$ as predicted by Eq. (41). It is also interesting to note that both EGC and SC achieve a better performance for the larger K_I.

In Fig. 5, we consider the SER of 8–PSK in Rayleigh fading with EGC. We assume that the composite noise impairing the received signal is the sum of two Rayleigh faded BPSK interferers [cf. Example E5) in Section 2.4] and Gaussian noise, where the interference power is 10 dB higher than the Gaussian noise power. For comparison we also consider the case of purely Gaussian noise. As expected from Eq. (17) for $L = 1$ both types of noise yield the same asymptotic SER. In contrast, assuming identical SNRs for $L = 2$ and $L = 3$ purely Gaussian noise is less favorable than the composite noise.

6 Conclusions

In this paper, we have presented a powerful new approach to the asymptotic SER analysis of linearly modulated signals impaired by fading and (possibly) non–Gaussian noise. Thereby, the only major assumption on the considered noise is that the Mellin transform $M_z(s)$ of its pdf exists for $s \geq 1$, which is true for most practically relevant types of noise. Based on this assumption we have provided general and simple–to–evaluate SER approximations for linear modulation formats with single–branch reception, EGC, and SC and for BOM, which become tight for high SNR. Our analysis has shown that the diversity gain is independent of the noise statistic and only depends on the fading statistic and the number of diversity branches. In contrast, the combining gain depends on both the type of fading and the type of noise. Therefore, in a log–log scale for high SNR the SER curves for different types of noise are all parallel and their relative shift depends on the Mellin transform of the noise pdfs.

A The Mellin Transform

The Mellin transform of the pdf $p_z(z)$ of the noise z plays a central role in this paper. Therefore, we discuss the Mellin transform and its properties in some detail in this appendix. For further reading we recommend [19, 20].

A.1 Definition and Existence

The Mellin transform $M_z(s) \triangleq \mathcal{M}\{p_z(z)\}$ of $p_z(z)$ is defined as

$$M_z(s) \triangleq \int_0^\infty z^{s-1} p_z(z)\,\mathrm{d}z \tag{42}$$

where both $p_z(z)$ and s may be complex in general. The Mellin transform exists if $\int_0^\infty z^{\Re\{s\}-1}|p_z(z)|\,\mathrm{d}z$ is finite. The interval $\alpha_l \leq \Re\{s\} \leq \alpha_u$ for which $M_z(s)$ exists is referred to as the *fundamental strip*. Tables of Mellin transforms can be found in [19].

In the remainder of this appendix, $p_z(z)$, $p_{z_1}(z)$, and $p_{z_2}(z)$ denote the pdfs of z, z_1, and z_2, respectively. Furthermore, we use the notations $M_z(s) \triangleq \mathcal{M}\{p_z(z)\}$, $M_{z_1}(s) \triangleq \mathcal{M}\{p_{z_1}(z)\}$, and $M_{z_2}(s) \triangleq \mathcal{M}\{p_{z_2}(z)\}$.

A.2 Basic Properties

The Mellin transform has many useful properties. A detailed discussion of these properties in the context of RVs and pdfs can be found in [20]. Here, we only state the properties most relevant to this paper without proof.

1) Scaling:

$$\mathcal{M}\{p_z(\alpha z)\} = \alpha^{-s} M_z(s), \qquad \alpha > 0. \tag{43}$$

2) Linearity:

$$\mathcal{M}\{\alpha_1 p_{z_1}(z) + \alpha_2 p_{z_2}(z)\} = \alpha_1 M_{z_1}(s) + \alpha_2 M_{z_2}(s). \tag{44}$$

3) Scaling of the RV: Let $z_1 = \alpha z$, $\alpha > 0$, then

$$M_{z_1}(z) = \alpha^{s-1} M_z(s). \tag{45}$$

A.3 Product of Two Independent RVs

The Mellin transform in Eq. (42) is only defined for positive z, whereas $p_z(z)$ is an even function of z, cf. AS1) in Section 2.3. Usually this discrepancy is not a problem and we can just ignore $p_z(z)$ for $z < 0$ when calculating the Mellin transform. However, care must be taken when calculating the Mellin transform of the product of two RVs.

Let $z = z_1 z_2$, where z_1 and z_2 are two independent RVs with even pdfs, respectively. The pdf of z can be expressed as $p_z(z) = \int_{-\infty}^{\infty} p_{z_1}(z_1) p_{z_2}(z/z_1)\, \mathrm{d}z_1/|z_1|$. Using the definition in Eq. (42) and exploiting $p_{z_1}(z) = p_{z_1}(-z)$ and $p_{z_2}(z) = p_{z_2}(-z)$, it is easy to show that the Mellin transform of $p_z(z)$ can be expressed as

$$M_z(s) = 2\, M_{z_1}(s)\, M_{z_2}(s) \tag{46}$$

which differs from [20, Eq. (15)] by a factor of two since in [20] non–negative RVs z_1, z_2 were assumed.

A.4 Sum of Two Independent RVs

Let $z = z_1 + z_2$, where z_1 and z_2 are independent RVs. For general s it is difficult to find a simple relation between $M_z(s)$ and $M_{z_1}(s)$, $M_{z_2}(s)$. In the following, we show however that such a relation exists if s is an integer.

First, we establish a relation between $M_z(s)$ and the moments $m_z(s)$ of z. If s is an even integer, we obtain

$$2M_z(s+1) = m_z(s) \triangleq \int\limits_{-\infty}^{\infty} z^s \, p_z(z) \, \mathrm{d}z. \tag{47}$$

Similarly, if s is odd, we can express $2M_z(s+1)$ as

$$2M_z(s+1) = \tilde{m}_z(s) \triangleq \int\limits_{-\infty}^{\infty} z^s \, \tilde{p}_z(z) \, \mathrm{d}z \tag{48}$$

where $\tilde{p}_z(z) = p(z)$, $z \geq 0$, and $\tilde{p}_z(z) = -p(z)$, $z < 0$. Note that $m_z(s) = 0$ and $\tilde{m}_z(s) = 0$ for odd and even s, respectively.

Recall that the Laplace transform of the pdf of z is given by $\Phi_z(p) = \Phi_{z_1}(p) \, \Phi_{z_2}(p)$, where $\Phi_{z_1}(p) \triangleq \mathcal{L}\{p_{z_1}(z)\}$ and $\Phi_{z_2}(p) \triangleq \mathcal{L}\{p_{z_2}(z)\}$.[9] For even s, the moments $m_z(s)$ of z can be calculated from

$$\begin{aligned} m_z(s) &= (-1)^s \frac{\mathrm{d}^s}{\mathrm{d}p^s} \Phi_z(p) \Big|_{p=0} = (-1)^s \sum_{k=0}^{s} \binom{s}{k} \frac{\mathrm{d}^{s-k}}{\mathrm{d}p^{s-k}} \Phi_{z_1}(p) \, \frac{\mathrm{d}^k}{\mathrm{d}p^k} \Phi_{z_2}(p) \Big|_{p=0} \\ &= (-1)^s \sum_{k=0}^{s} \binom{s}{k} m_{z_1}(s-k) m_{z_2}(k) \end{aligned} \tag{49}$$

where $m_{z_1}(s)$ and $m_{z_2}(s)$ denote the sth moment of z_1 and z_2, respectively. It is easy to show that Eq. (49) also holds for odd s if $\tilde{m}_{(\cdot)}$ is replaced by $m_{(\cdot)}$. Therefore, taking Eqs. (47) and (48) into account, the Mellin transforms of $p_z(z)$ for integer s can be calculated from

$$M_z(s) = \begin{cases} 2 \sum_{k=0}^{(s-1)/2} \binom{s-1}{2k} M_{z_1}(s-2k) M_{z_2}(2k+1), & s \text{ odd} \\ 2 \sum_{k=0}^{s/2-1} \binom{s-1}{2k+1} M_{z_1}(s-2k) M_{z_2}(2k+2), & s \text{ even} \end{cases}, \quad s \geq 1. \tag{50}$$

[9] Following the literature, in general we use "s" as transformation variable for both Mellin and Laplace transform. In this appendix, however, we deviate from this practice to avoid confusion and use "p" as transformation variable for the Laplace transform.

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Figures:

Table 1: Pdf $p_a(a)$ of fading amplitude $a \geq 0$ and corresponding series parameters p_k, ξ, and δ for the series expansion $p_a(a) = \frac{1}{a} \sum_{k=1}^{\infty} p_k \, (a^2/\bar{\gamma})^{\xi k + \delta}$. $\bar{\gamma} = \mathcal{E}\{a^2\}$.

Fading Model	Pdf of Fading Amplitude a and Series Parameters
Rayleigh	$p_a(a) = \frac{2a}{\bar{\gamma}} \exp\left(-\frac{a^2}{\bar{\gamma}}\right)$ $p_k = \frac{2(-1)^{k-1}}{(k-1)!}$ $\xi = 1, \quad \delta = 0$
Ricean $K \geq 0$	$p_a(a) = \frac{2(K+1)a}{\bar{\gamma}} \exp\left(-K - \frac{(1+K)a^2}{\bar{\gamma}}\right) \mathrm{I}_0\left(2a\sqrt{\frac{K(K+1)}{\bar{\gamma}}}\right)$ $p_k = 2(K+1)^k e^{-K} \sum_{\kappa=0}^{k-1} \frac{(-1)^{k-1-\kappa} K^{\kappa}}{(k-1-\kappa)!\,(\kappa!)^2}$ $\xi = 1, \quad \delta = 0$
Nakagami–q $b = \frac{1-q^2}{1+q^2}$ $0 \leq q < 1$	$p_a(a) = \frac{2a}{\bar{\gamma}\sqrt{1-b^2}} \exp\left(-\frac{a^2}{(1-b^2)\bar{\gamma}}\right) \mathrm{I}_0\left(\frac{ba^2}{(1-b^2)\bar{\gamma}}\right)$ $p_k = \frac{2}{(1-b^2)^{k+1/2}} \sum_{\kappa=0}^{\lfloor k/2 \rfloor} \frac{(-1)^{k-2\kappa}(b/2)^{2\kappa}}{(\kappa!)^2\,(k-2\kappa)!}$ $\xi = 1, \quad \delta = 0$
Nakagami–m $m \geq 1/2$	$p_a(a) = \frac{2}{\Gamma(m)} \left(\frac{m}{\bar{\gamma}}\right)^m a^{2m-1} \exp\left(-\frac{ma^2}{\bar{\gamma}}\right)$ $p_k = \frac{2(-1)^{k-1} m^{k+m-1}}{\Gamma(m)(k-1)!}$ $\xi = 1, \quad \delta = m-1$
Weibull $c > 0$	$p_a(a) = c\left(\frac{\Gamma(1+2/c)}{\bar{\gamma}}\right)^{c/2} a^{c-1} \exp\left[-\left(\frac{a^2}{\bar{\gamma}}\Gamma(1+2/c)\right)^{c/2}\right]$ $p_k = \frac{(-1)^{k-1}(\Gamma(1+2/c))^{ck/2}}{(k-1)!}$ $\xi = c/2, \delta = 0$

Table 2: Pdf $p_z(z)$, Laplace transform $\Phi_z(s) = \mathcal{L}\{p_z(z)\}$, and Mellin transform $M_z(s) = \mathcal{M}\{p_z(z)\}$ for different types of noise. Noise variance $\sigma_z^2 = 1$ in all cases. Generalized Gaussian noise: $\eta(C) = [\Gamma(1/C)/\Gamma(3/C)]^{C/2}$. BPSK interference with fixed channel phase (CP): $\mathcal{S}$ is the set of all 2^I possible sums of the $\pm d_k$, $1 \le k \le I$. $\mathcal{S}^+$ contains all positive elements of $\mathcal{S}$. We note that for generalized Gaussian noise a closed–form expression for $\Phi_z(s)$ does not seem to exist for general $C > 0$.

Noise Model	$p_z(z)$, $\Phi_z(s)$, and $M_z(s)$
Gaussian Noise	$p_z(z) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{z^2}{2}\right)$ $\Phi_z(s) = e^{s^2/2}$ $M_z(s) = \frac{1}{2\sqrt{2\pi}} \Gamma\left(\frac{s}{2}\right) 2^{\frac{s}{2}}$
Generalized Gaussian Noise, $C > 0$	$p_z(z) = \frac{C}{2\,\Gamma(1/C)(\eta(C))^{1/C}} \exp\left(-\frac{\lvert z\rvert^C}{\eta(C)}\right)$ $M_z(s) = \frac{1}{2\Gamma(1/C)} \Gamma\left(\frac{s}{C}\right) (\eta(C))^{\frac{s-1}{C}}$
Gaussian Mixture $\sum_{k=1}^{I} c_k = 1$ $\sum_{k=1}^{I} c_k \sigma_{z_k}^2 = 1$	$p_z(z) = \sum_{k=1}^{I} \frac{c_k}{\sqrt{2\pi\sigma_{z_k}^2}} \exp\left(-\frac{z^2}{2\sigma_{z_k}^2}\right)$ $\Phi_z(s) = \sum_{k=1}^{I} c_k e^{s^2\sigma_{z_k}^2/2}$ $M_z(s) = \frac{\Gamma\left(\frac{s}{2}\right) 2^{\frac{s}{2}}}{2\sqrt{2\pi}} \sum_{k=1}^{I} c_k\, \sigma_{z_k}^{s-1}$
BPSK Interference with Fixed CP $\sum_{k=1}^{I} \lvert d_k\rvert^2 = 1$	$p_z(z) = \frac{1}{2^I} \sum_{\bar{d}\in\mathcal{S}} \delta(z - \bar{d})$ $\Phi_z(s) = \frac{1}{2^I} \sum_{\bar{d}\in\mathcal{S}} e^{-s\bar{d}}$ $M_z(s) = \frac{1}{2^I} \sum_{\bar{d}\in\mathcal{S}^+} \bar{d}^{s-1}$
M–PSK Interference with Random CP $\sum_{k=1}^{I} \lvert d_k\rvert^2 = 2$	$p_z(z) = \frac{1}{\pi\sqrt{\lvert d_1\rvert^2 - z^2}}$, $\lvert z\rvert < \lvert d_1\rvert$, $I = 1$ $\Phi_z(s) = \prod_{k=1}^{I} \mathrm{I}_0(\lvert d_k\rvert s)$ $M_z(s) = \frac{\lvert d_1\rvert^{s-1}\Gamma\left(\frac{s}{2}\right)}{2\sqrt{\pi}\Gamma\left(\frac{s+1}{2}\right)}$, $I = 1$

Table 3: Parameters β_M^{X} and d_M^{X} for M–ary modulation schemes X $\in$ {PAM, PSK, QAM, BOM}.

Modulation Scheme	β_M^{X}	d_M^{X}
M–PAM	$1 - \frac{1}{M}$	$\sqrt{\frac{6}{M^2-1}}$
BPSK ($M = 2$)	$\frac{1}{2}$	$\sqrt{2}$
M–PSK ($M \geq 4$)	1	$\sqrt{2}\sin\left(\frac{\pi}{M}\right)$
M–QAM	$2\left(1 - \frac{1}{\sqrt{M}}\right)$	$\sqrt{\frac{3}{M-1}}$
BOM ($M = 2$)	$\frac{1}{2}$	$\sqrt{2}$

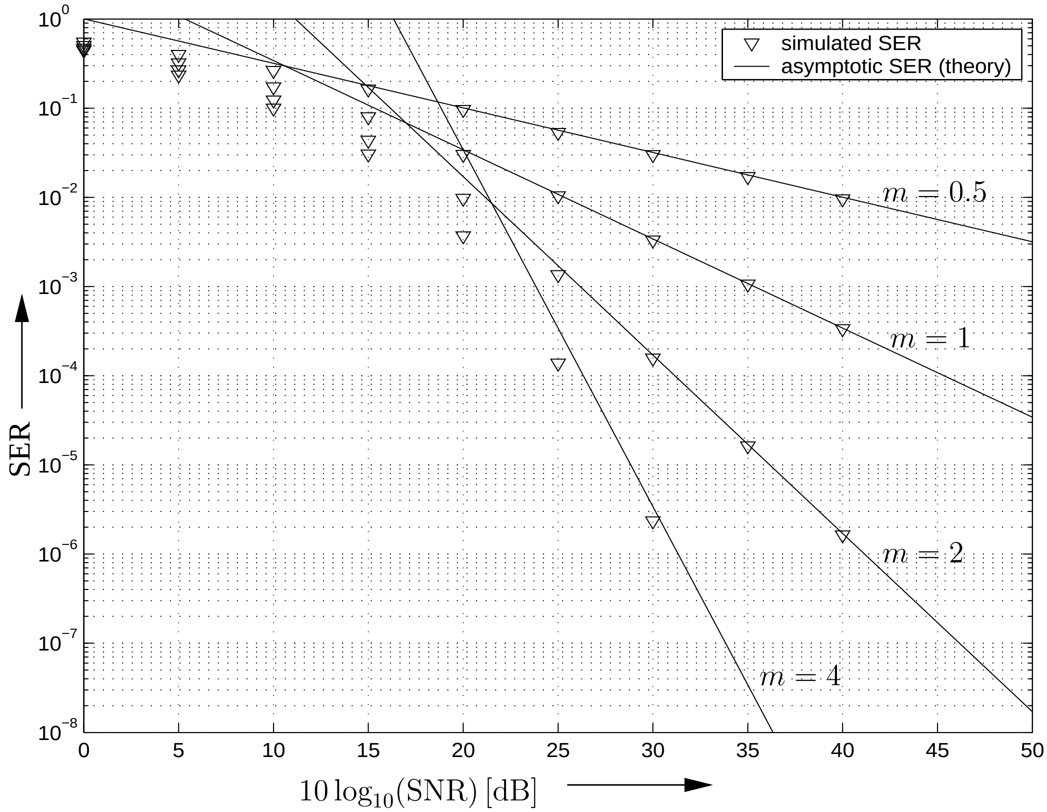

Figure 1: SER vs. SNR for 8–PSK over a Nakagami–m fading channel with ϵ–mixture noise ($\epsilon = 0.25$, $\kappa = 10$). Markers: Simulated SER. Solid lines: Asymptotic SER [Eq. (18)].

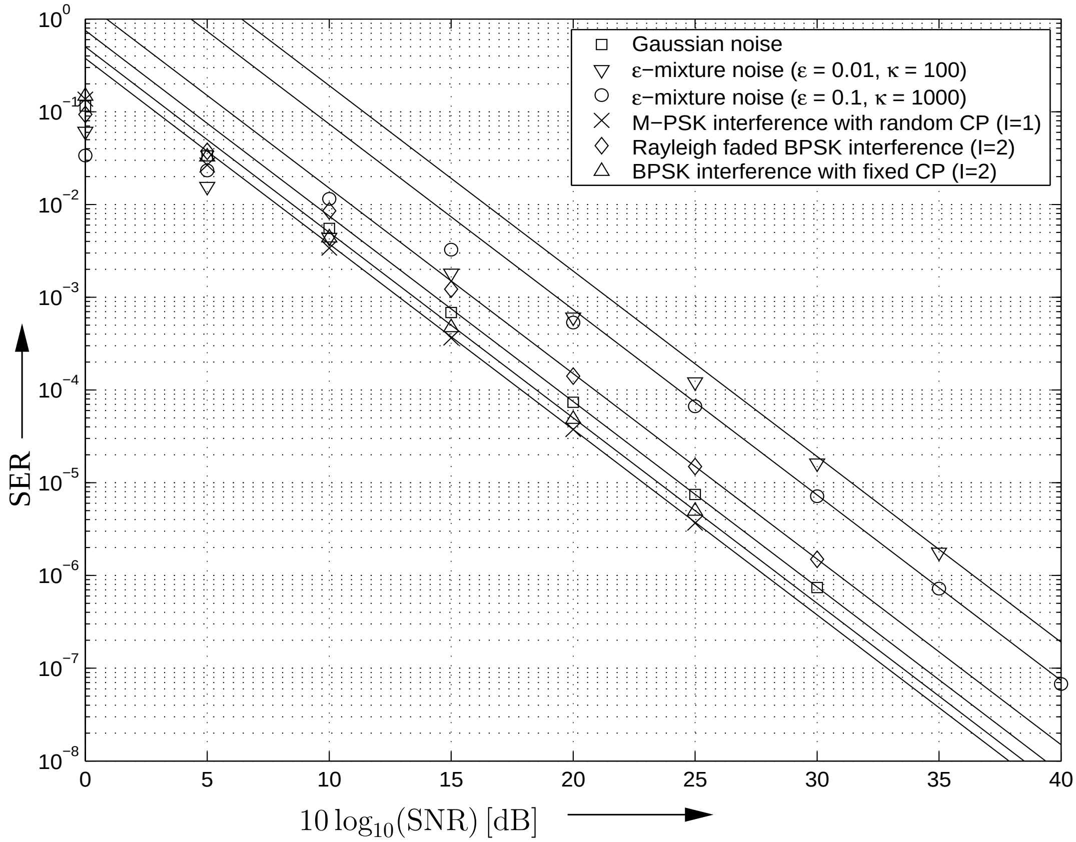

Figure 2: SER vs. SNR for BPSK over a Nakagami–m fading channel with $m = 2$ and different types of noise discussed in Sections 2.3, 2.4. ϵ–mixture noise: Example E1) in Section 2.3. M–PSK interference with random channel phase (CP): Example E3) in Section 2.3. Rayleigh faded BPSK interference: Example E5) in Section 2.4. BPSK interference with fixed CP: Example E2) in Section 2.3. Markers: Simulated SER. Solid lines: Asymptotic SER [Eq. (18)].

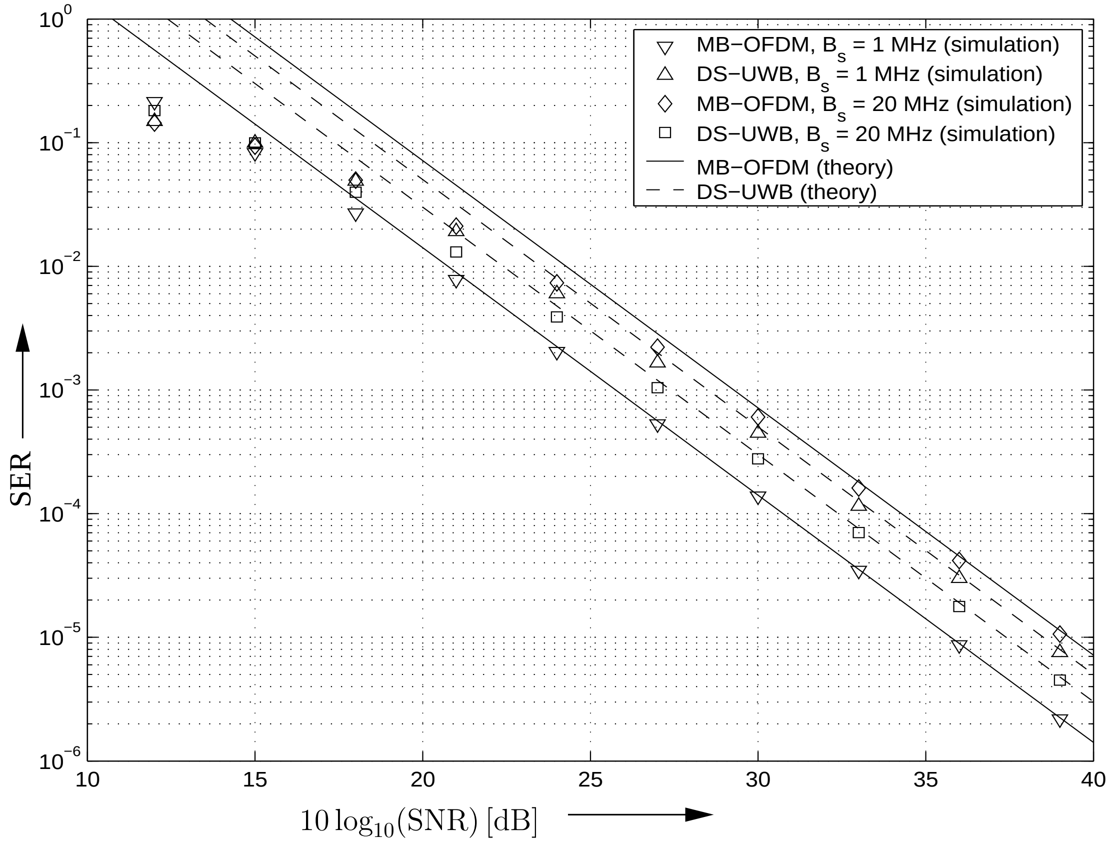

Figure 3: SER vs. SNR for 16–QAM with bandwidth B_s over a Nakagami–m fading channel with $m = 2$ and UWB interference. Markers: Simulated SER. Solid lines: Asymptotic SER for MB–OFDM interference [Eq. (18)]. Dashed lines: Asymptotic SER for DS–UWB interference [Eq. (18)].

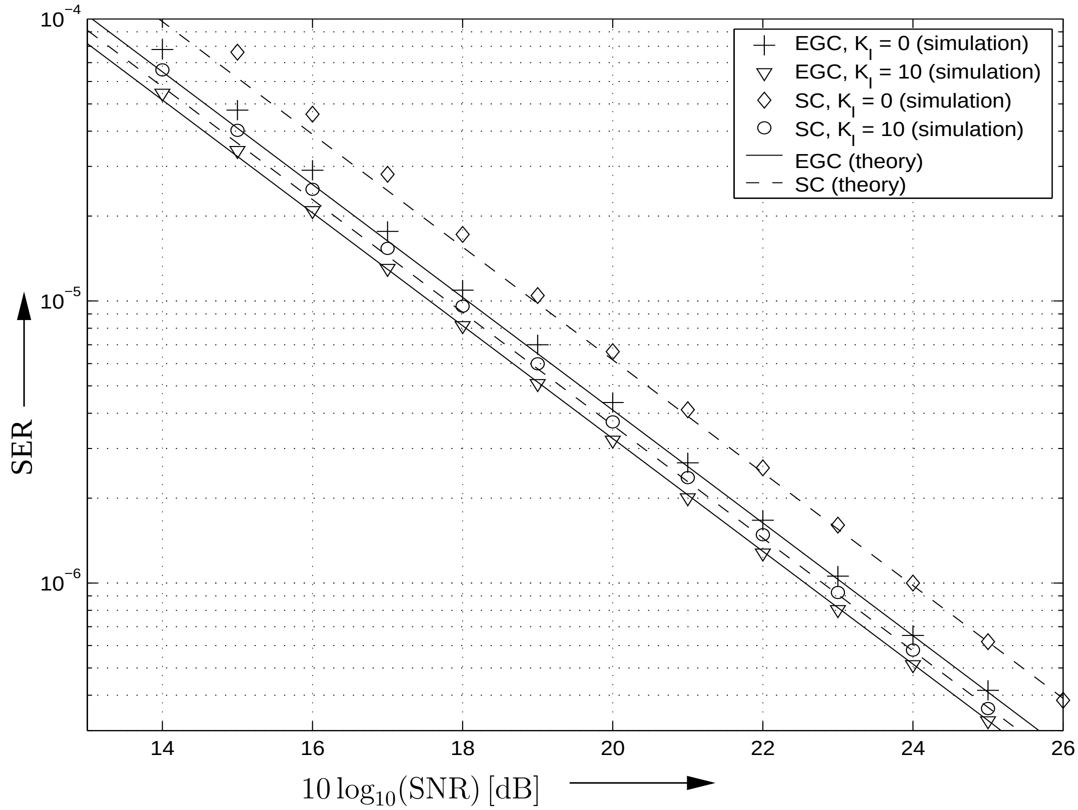

Figure 4: SER vs. SNR per branch for BPSK over a Ricean fading channel with Ricean factor $K = 2$, $L = 2$ diversity branches, and Ricean faded M–PSK interference. The interference channel has Ricean factors of $K_I = 0$ and $K_I = 10$, respectively. All diversity paths have the same average SNR. EGC and SC are considered. Markers: Simulated SER. Solid lines: Asymptotic SER for EGC [Eq. (31)]. Dashed lines: Asymptotic SER for SC [Eq. (36)].

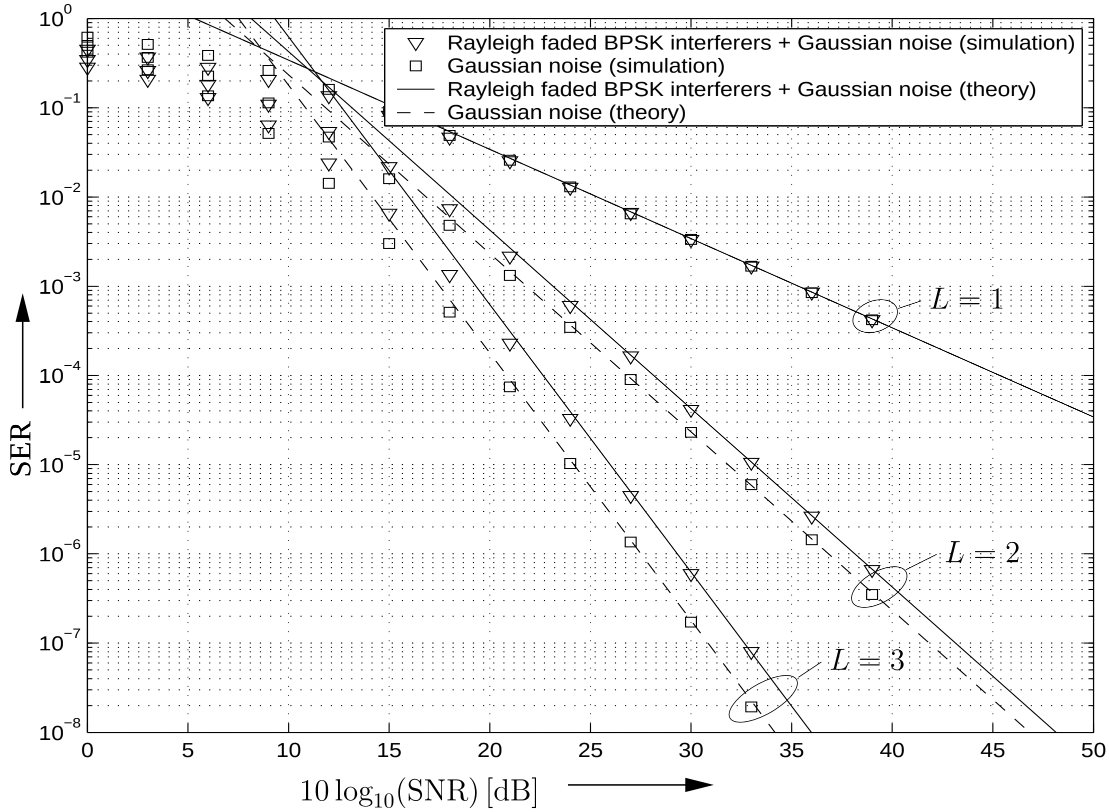

Figure 5: SER vs. SNR per branch for 8–PSK over a Rayleigh fading channel with EGC. All diversity paths have the same average SNR. Markers: Simulated SER. Solid lines: Asymptotic SER for Rayleigh faded BPSK interference (two interferers) and Gaussian noise [Eq. (31)]. Dashed lines: Asymptotic SER for Gaussian noise [Eq. (31)].