

# Error Rate Performance of Network–Coded Cooperative Diversity Systems

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**Abstract**—In this paper, we study network–coded cooperative diversity (NCCD) systems comprising multiple sources, one relay, and one destination. The relay detects the packets received from all sources and performs Galois field network coding. We propose a simple cooperative maximum–ratio combining scheme for the destination which is shown to achieve the maximum diversity gain of the system. Furthermore, we provide a mathematical framework for the asymptotic analysis of NCCD systems with  $M$ –ary modulation for high signal–to–noise ratios. Based on this framework, we derive simple and elegant closed–form expressions for the asymptotic symbol and bit error rates which provide significant insight into the impact of various system and channel parameters on performance and can be exploited for performance optimization. Simulation results confirm the accuracy of the presented analysis and show that large performance gains are possible by optimizing the power allocation in NCCD systems based on the developed analytical results.

## I. INTRODUCTION

Cooperative diversity (CD) is an effective technique to exploit the spatial diversity offered by wireless relay nodes. The main drawback of CD schemes is a reduction in throughput since the different cooperating terminals use orthogonal channels for transmission [1]. This throughput reduction is most noticeable in CD systems with multiple source terminals, since the relays forward the signals received by each source in a separate time slot or frequency band.

One effective approach to increase the throughput in multi–source CD systems is network coding [2]–[5]. The idea of network coding was originally developed for wired networks as an efficient routing technique capable of enhancing the network throughput [6]. However, network coding also allows a relay to first encode the packets received by several sources before forwarding a single encoded packet to the destination. Thus, the relay can simultaneously serve multiple sources and the network throughput is substantially increased.

The performance of the combination of CD and network coding, which is referred to as network–coded CD (NCCD), has been studied recently in the literature. In particular, the outage capacity and the diversity–multiplexing tradeoff of such a system was analyzed in [3] and [4], respectively, and its outage probability was calculated in [2]. Common to all these works is the assumption of error–free source–relay channels. Although this assumption greatly simplifies the analysis of NCCD systems, it may not be valid in practical wireless networks where detection errors at the relay may be caused by fading and noise. We also note that previous work on NCCD has focused on network coding over the Galois field (GF) of order two limiting the adopted modulation schemes to binary. Furthermore, a general and accurate error rate analysis giving insight into the performance of NCCD systems is not available in the literature.

In this paper, we consider an NCCD system with multiple sources using general  $M$ –ary modulation schemes, one

relay, and one destination. We propose a novel cooperative maximum–ratio combining (C–MRC) scheme at the destination, which guarantees full diversity for all sources even if the non–ideal detection at the relay is taken into account. The proposed C–MRC scheme may be viewed as a generalization of a similar scheme that was proposed for CD for a single source in [7]. Furthermore, we derive simple and elegant closed–form expressions for the asymptotic symbol and bit error rates of NCCD with C–MRC in Rayleigh fading. These closed–form expressions give valuable insight into the impact of various system and channel parameters such as the number of sources and the signal–to–noise ratios (SNRs) of the involved wireless channels. For example, our analytical results reveal that the diversity gain for all source terminals is two irrespective of the number of sources but the coding gain decreases as the number of sources increases. The derived error rate expressions can also be exploited for various NCCD system optimization problems such as optimal power allocation, relay selection, and relay placement.

The remainder of this paper is organized as follows. In Section II, the system model for the considered NCCD system as well as some notations and definitions are introduced. Asymptotic expressions for the symbol error rate (SER) and the bit error rate (BER) are derived in Section III. Numerical and simulation results are presented in Section IV, and conclusions are drawn in Section V.

## II. PRELIMINARIES

In this section, we describe the model for the considered NCCD system and introduce some notations and definitions.

### A. Notations and Definitions

In this paper,  $[\cdot]^T$ ,  $(\cdot)^*$ ,  $\Re\{\cdot\}$ ,  $\mathcal{E}_x\{\cdot\}$ , and  $\Gamma(\cdot)$ , denote transposition, complex conjugation, the real part of a complex number, statistical expectation with respect to  $x$ , and the Gamma function, respectively.  $Q(x) \triangleq \frac{1}{\sqrt{2\pi}} \int_x^\infty e^{-t^2/2} dt$  denotes the Gaussian  $Q$ –function. Furthermore, we use the notation  $u \doteq v$  to indicate that  $u$  and  $v$  are asymptotically equivalent, and a function  $f(x)$  is  $o(g(x))$  if  $\lim_{x \rightarrow 0} f(x)/g(x) = 0$ .

### B. Signal Model

The considered NCCD system is depicted in Fig. 1 and comprises  $N_s$  source terminals  $S_i$ ,  $1 \leq i \leq N_s$ , one relay  $R$ , and one destination terminal  $D$ . Transmission from the source terminals to the destination terminal is organized in two hops. The first hop comprises  $N_s$  time slots and each source terminal  $S_i$ ,  $1 \leq i \leq N_s$ , transmits a data packet to the relay and the destination using one time slot. In particular, the data symbol  $s_i \in \mathcal{A}$  is generated at the source  $S_i$ , where  $\mathcal{A} \triangleq \text{GF}(2^m)$  is the GF of order  $M = 2^m$ . Data symbol  $s_i$  is mapped to a transmit symbol  $x_i \in \mathcal{X}$  with  $\mathcal{E}\{|x_i|^2\} = 1$  using

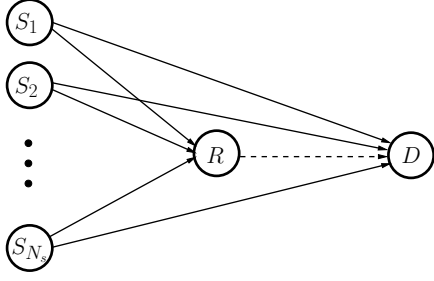


Fig. 1. Block diagram of the considered NCCD system. Solid and dashed lines denote links belonging to first and second hop, respectively.

the mapping  $x_i = \mu_{\mathcal{X}}(s_i)$ , where  $\mathcal{X}$  denotes an  $M$ -ary signal constellation such as  $M$ -ary phase-shift keying ( $M$ -PSK) and  $\mu_{\mathcal{X}} : \mathcal{A} \rightarrow \mathcal{X}$  is a one-to-one mapping function from  $\mathcal{A}$  to  $\mathcal{X}$ . The transmit symbols  $x_i$  are transmitted to the relay and the destination. The signals received by the destination and the relay in the first hop are given by

$$r_{S_i D} = \sqrt{P_i} f_i x_i + n_{D,i}, \quad 1 \leq i \leq N_s, \quad (1)$$

and

$$r_{S_i R} = \sqrt{P_i} g_i x_i + n_{R,i}, \quad 1 \leq i \leq N_s, \quad (2)$$

respectively, where  $P_i$  is the average transmit power of the  $i$ th source, and  $f_i$  and  $g_i$  denote the fading gains of the  $S_i \rightarrow D$  and the  $S_i \rightarrow R$  channels, respectively. Furthermore,  $n_{D,i}$  and  $n_{R,i}$  denote the additive white Gaussian noise (AWGN) samples at the destination and the relay, respectively. The variances of these noise samples are denoted by  $\sigma_{n_{D,i}}^2 \triangleq \mathcal{E}\{|n_{D,i}|^2\}$  and  $\sigma_{n_{R,i}}^2 \triangleq \mathcal{E}\{|n_{R,i}|^2\}$ , respectively.

Having received the signals  $r_{S_i R}$  the relay performs coherent maximum-likelihood (ML) detection to obtain the detected symbols

$$\hat{x}_{R,i} = \arg \min_{\tilde{x} \in \mathcal{X}} \{|r_{S_i R} - \sqrt{P_i} g_i \tilde{x}|^2\}, \quad 1 \leq i \leq N_s. \quad (3)$$

The corresponding detected data symbol is given by  $\hat{s}_{R,i} = \mu_{\mathcal{X}}^{-1}(\hat{x}_{R,i})$ .

The second hop comprises a single time slot. In particular, the relay performs network coding and computes the data symbol  $\hat{s}_R = \hat{s}_{R,1} \oplus \dots \oplus \hat{s}_{R,N_s}$ , where  $\oplus$  denotes addition in  $\text{GF}(2^m)$ . The relay forwards transmit symbol  $\hat{x}_R = \mu_{\mathcal{X}}(\hat{s}_R)$  to the destination. The signal received at the destination in the second hop,  $r_{RD}$ , can be modeled as

$$r_{RD} = \sqrt{P_R} h_R \hat{x}_R + n_{D,R}, \quad (4)$$

where  $P_R$  is the average transmit power of the relay,  $h_R$  is the fading gain of the  $R \rightarrow D$  channel, and  $n_{D,R}$  is the AWGN at the destination in the second hop having variance  $\sigma_{n_{D,R}}^2 \triangleq \mathcal{E}\{|n_{D,R}|^2\}$ .

Throughout this paper we assume independent Rayleigh fading for all links of the network. Thus, the fading gains  $f_i \triangleq a_{f_i} e^{-j\theta_{f_i}}$ ,  $h_i \triangleq a_{g_i} e^{-j\theta_{g_i}}$ ,  $1 \leq i \leq N_s$ , and  $h_R \triangleq a_{h_R} e^{-j\theta_{h_R}}$ , are independent Gaussian random variables (RVs) with zero mean and variances  $\Omega_{f_i} \triangleq \mathcal{E}\{|f_i|^2\}$ ,  $\Omega_{g_i} \triangleq \mathcal{E}\{|g_i|^2\}$ ,  $1 \leq i \leq N_s$ , and  $\Omega_R \triangleq \mathcal{E}\{|h_R|^2\}$ , respectively. Here, the channel amplitudes  $a_{f_i}$ ,  $a_{g_i}$ , and  $a_{h_R}$  are positive real RVs and follow a Rayleigh distribution. Furthermore, the channel phases  $\theta_{f_i}$ ,  $\theta_{g_i}$ , and  $\theta_{h_R}$  are uniformly distributed in  $[-\pi, \pi)$  and are independent from the channel amplitudes.

For future reference, we define the instantaneous SNRs of the  $S_i \rightarrow D$ ,  $S_i \rightarrow R$ , and  $R \rightarrow D$  links as  $\gamma_{f_i} \triangleq$

$P_i a_{f_i}^2 / \sigma_{n_{D,i}}^2$ ,  $\gamma_{g_i} \triangleq P_i a_{g_i}^2 / \sigma_{n_{R,i}}^2$ , and  $\gamma_{h_R} \triangleq P_R a_{h_R}^2 / \sigma_{n_{D,R}}^2$ , respectively. The corresponding average SNRs are given by  $\bar{\gamma}_{f_i} = P_i \Omega_{f_i} / \sigma_{n_{D,i}}^2$ ,  $\bar{\gamma}_{g_i} = P_i \Omega_{g_i} / \sigma_{n_{R,i}}^2$ , and  $\bar{\gamma}_{D,R} = P_R \Omega_R / \sigma_{n_{D,R}}^2$ , respectively.

**Remark 1:** Based on the presented signal model, a total of  $N_s + 1$  time slots are required for the transmission of signals from all sources to the destination. In contrast, a conventional CD system [1], [7] requires  $2N_s$  time slots since the relay assists only a single source at a time.

### C. Equivalent Source-Relay Channel

Similar to conventional CD [7], it is also convenient for NCCD to introduce an equivalent channel between the source terminals and the relay. This will be particularly useful for the diversity combining scheme in Section II-D and the performance analysis in Section III. The input of this equivalent channel is the relay transmit symbol in the absence of noise,  $x_R \triangleq \mu_{\mathcal{X}}(\mu_{\mathcal{X}}^{-1}(x_1) \oplus \dots \oplus \mu_{\mathcal{X}}^{-1}(x_{N_s})) \in \mathcal{X}$  and the output is the actual relay transmit symbol,  $\hat{x}_R \in \mathcal{X}$ . Defining the source-relay SNR vector  $\gamma_g \triangleq [\gamma_{g_1}, \dots, \gamma_{g_{N_s}}]^T$ , this channel is characterized by the equivalent error probability  $P_{e,\text{eq}}(\gamma_g)$ , which is the probability that  $\hat{x}_R \neq x_R$ . For an  $M$ -ary signal constellation  $\mathcal{X}$ , the equivalent error probability  $P_{e,\text{eq}}(\gamma_g)$  is given by  $P_{e,\text{eq}}(\gamma_g) = \beta Q(\sqrt{2\alpha\gamma_{\text{eq}}(\gamma_g)})$ , where  $\alpha$  and  $\beta$  are two modulation dependent constants (e.g.  $\alpha = \beta = 1$  for BPSK) and  $\gamma_{\text{eq}}(\gamma_g)$  is the instantaneous SNR associated with the equivalent source-relay channel. This equivalent SNR can be expressed as

$$\gamma_{\text{eq}}(\gamma_g) = \frac{1}{2\alpha} (Q^{-1}(P_{e,\text{eq}}(\gamma_g)/\beta))^2. \quad (5)$$

It can be shown that for sufficiently high SNR (please refer to [8] for a proof)  $\gamma_{\text{eq}}(\gamma_g)$  can be accurately approximated as

$$\gamma_{\text{eq}}(\gamma_g) = \min\{\gamma_{g_1}, \dots, \gamma_{g_{N_s}}\}. \quad (6)$$

We note that since  $\gamma_{g_i}$ ,  $1 \leq i \leq N_s$ , is an exponentially distributed RV with mean  $\bar{\gamma}_{g_i}^{-1}$ ,  $\gamma_{\text{eq}}(\gamma_g)$  in (6) is also exponentially distributed with mean  $\bar{\gamma}_{\text{eq}}^{-1} = 1/\bar{\gamma}_{g_1} + \dots + 1/\bar{\gamma}_{g_{N_s}}$ .

### D. Diversity Combining at the Destination

Due to the possibly erroneous decisions at the relay, the ML decision metric at the destination is highly complex and not amenable to analysis. In order to avoid the problems associated with the ML metric, we extend the C-MRC scheme proposed in [7] for conventional CD to NCCD. As will be shown in Section III, the simple C-MRC scheme achieves the full diversity of NCCD systems with any number of sources. The corresponding decision rule is given by

$$\hat{x}_D = \arg \min_{\tilde{x} \in \mathcal{X}^{N_s}} \{m_c(\tilde{x})\}, \quad (7)$$

where vector  $\hat{x}_D \triangleq [\hat{x}_{D,1} \dots \hat{x}_{D,N_s}]^T$  contains the detected symbols at the destination for all sources, vector  $\tilde{x} \triangleq [\tilde{x}_1 \dots \tilde{x}_{N_s}]^T \in \mathcal{X}^{N_s}$  contains trial transmit symbols  $\tilde{x}_i \in \mathcal{X}$ ,  $1 \leq i \leq N_s$ , and  $m_c(\tilde{x})$  is the C-MRC metric. The decoded data symbols are obtained as  $\hat{s}_{D,i} = \mu_{\mathcal{X}}^{-1}(\hat{x}_{D,i})$ ,  $1 \leq i \leq N_s$ , and the C-MRC metric is given by

$$m_c(\tilde{x}) = \sum_{i=1}^{N_s} \frac{|r_{S_i D} - \sqrt{P_i} f_i \tilde{x}_i|^2}{\sigma_{n_{D,i}}^2} + \lambda_R \frac{|r_{RD} - \sqrt{P_R} h_R \tilde{x}_R|^2}{\sigma_{n_{D,R}}^2} \quad (8)$$

where  $\tilde{x}_R \triangleq \mu_{\mathcal{X}}(\mu_{\mathcal{X}}^{-1}(\tilde{x}_1) \oplus \dots \oplus \mu_{\mathcal{X}}^{-1}(\tilde{x}_{N_s}))$ , and  $\lambda_R$  is a weighting factor which is defined as

$$\lambda_R \triangleq \frac{\min\{\gamma_{\text{eq}}(\gamma_g), \gamma_R\}}{\gamma_R}. \quad (9)$$

In order to compute  $\lambda_R$ , the receiver has to know the SNR of the weakest source-relay channel. This SNR can be measured at the relay and then forwarded to the destination over a low-rate feedback link. As mentioned before, the proposed C-MRC scheme is a generalization of the scheme in [7], which is obtained as a special case for  $N_s = 1$ , where NCCD reduces to conventional CD.

### III. ASYMPTOTIC PERFORMANCE ANALYSIS

In this section, we analyze the asymptotic error rate performance of the considered NCCD system for high SNRs, i.e.,  $\bar{\gamma}_{f_i}, \bar{\gamma}_{g_i} \rightarrow \infty$ ,  $1 \leq i \leq N_s$ , and  $\bar{\gamma}_R \rightarrow \infty$ . In particular, we develop asymptotic closed-form expressions for the (average) pairwise error probability (PEP), SER, and BER.

For convenience, we introduce the source-destination SNR vector  $\gamma_f \triangleq [\gamma_{f_1}, \dots, \gamma_{f_{N_s}}]^T$ , the normalized noise samples  $\bar{n}_{D,i} \triangleq n_{D,i}/\sigma_{n_{D,i}}$ ,  $1 \leq i \leq N_s$ , and  $\bar{n}_{D,R} \triangleq n_{D,R}/\sigma_{n_{D,R}}$ , and noise vector  $\mathbf{n} \triangleq [\bar{n}_{D,1}, \dots, \bar{n}_{D,N_s}, \bar{n}_{D,R}]^T$ .

#### A. Asymptotic Pairwise Error Probability

Assuming that  $\mathbf{x} \triangleq [x_1 \dots x_{N_s}]^T \in \mathcal{X}^{N_s}$  was transmitted by the sources and  $\tilde{\mathbf{x}} \triangleq [\tilde{x}_1 \dots \tilde{x}_{N_s}]^T \in \mathcal{X}^{N_s}$ ,  $\tilde{\mathbf{x}} \neq \mathbf{x}$ , was detected at the destination, the PEP for the considered NCCD system can be expressed as

$$P(\mathbf{x} \rightarrow \tilde{\mathbf{x}}) = \Pr\{m_c(\mathbf{x}) > m_c(\tilde{\mathbf{x}})\}. \quad (10)$$

It is convenient to first obtain the PEP conditioned on the instantaneous SNRs  $\gamma_f, \gamma_g, \gamma_R$ , and the noise vector  $\mathbf{n}$ . This conditional PEP can be expressed as

$$P(\mathbf{x} \rightarrow \tilde{\mathbf{x}} | \gamma_f, \gamma_g, \gamma_R, \mathbf{n}) = [1 - P_{e,\text{eq}}(\gamma_g)] P(\mathbf{x} \rightarrow \tilde{\mathbf{x}} | x_R, \gamma_f, \gamma_{\text{eq}}, \gamma_R, \mathbf{n}) + \frac{1}{|\mathcal{N}(x_R)|} \sum_{\hat{x}_R \in \mathcal{N}(x_R)} P_{e,\text{eq}}(\gamma_g) P(\mathbf{x} \rightarrow \tilde{\mathbf{x}} | \hat{x}_R, \gamma_f, \gamma_{\text{eq}}, \gamma_R, \mathbf{n}) \quad (11)$$

where  $x_R$  and  $P_{e,\text{eq}}(\gamma_g)$  have been defined in Subsection II-C. In deriving (11), we have assumed that the erroneous  $\hat{x}_R \in \mathcal{X}$  is a nearest neighbor of  $x_R$ , i.e.,  $\hat{x}_R \in \mathcal{N}(x_R)$ , where set  $\mathcal{N}(x_R)$  contains all nearest neighbors of  $x_R$  in  $\mathcal{X}$ . This approximation is well justified for  $\bar{\gamma}_{g_i} \rightarrow \infty$ ,  $1 \leq i \leq N_s$ , and its accuracy will be confirmed by simulations in Section IV. The conditional PEP  $P(\mathbf{x} \rightarrow \tilde{\mathbf{x}} | \bar{x}_R, \gamma_f, \gamma_{\text{eq}}, \gamma_R, \mathbf{n})$ ,  $\bar{x}_R \in \{x_R, \hat{x}_R\}$ , can be expressed as

$$P(\mathbf{x} \rightarrow \tilde{\mathbf{x}} | \bar{x}_R, \gamma_f, \gamma_{\text{eq}}, \gamma_R, \mathbf{n}) = \Pr\{m_c(\mathbf{x}) > m_c(\tilde{\mathbf{x}}) | \bar{x}_R, \gamma_f, \gamma_{\text{eq}}, \gamma_R, \mathbf{n}\} = \Pr\left\{\sum_{i=1}^{N_s} \Delta_{f_i}(x_i, \tilde{x}_i) + \lambda_R \Delta_R(x_R, \tilde{x}_R, \bar{x}_R) < 0 \middle| \gamma_f, \gamma_{\text{eq}}, \gamma_R, \mathbf{n}\right\} \quad (12)$$

where

$$\Delta_{f_i}(x_i, \tilde{x}_i) \triangleq |\sqrt{\gamma_{f_i}}(x_i - \tilde{x}_i) + \bar{n}_{D,i}|^2, \quad (13)$$

and

$$\Delta_R(x_R, \tilde{x}_R, \bar{x}_R) \triangleq |\sqrt{\gamma_R}(x_R - \bar{x}_R) + \bar{n}_{D,R}|^2 - |\sqrt{\gamma_R}(\tilde{x}_R - \bar{x}_R) + \bar{n}_{D,R}|^2. \quad (14)$$

For derivation of the unconditional PEP, we exploit that for any RV  $\Delta$  we have  $\Pr\{\Delta < 0\} = \frac{1}{2\pi j} \int_{c-j\infty}^{c+j\infty} \Phi_{\Delta}(s) \frac{ds}{s}$  with moment generating function (MGF)  $\Phi_{\Delta}(s) \triangleq \mathcal{E}_{\Delta}\{e^{-\Delta s}\}$  and  $P_{e,\text{eq}}(\gamma_g) = \beta Q(\sqrt{2\alpha\gamma_{\text{eq}}})$ , cf. Subsection II-C. Using these relations, we obtain the unconditional PEP from (11) and (12) as

$$P(\mathbf{x} \rightarrow \tilde{\mathbf{x}}) = \mathcal{E}_{\gamma_f, \gamma_g, \gamma_R, \mathbf{n}}\{P(\mathbf{x} \rightarrow \tilde{\mathbf{x}} | \gamma_f, \gamma_g, \gamma_R, \mathbf{n})\} = \frac{1}{2\pi j} \int_{c-j\infty}^{c+j\infty} \left(\prod_{i=1}^{N_s} \Phi_{f_i}(s)\right) \Phi_R(s) \frac{ds}{s}, \quad (15)$$

where  $c$  is a small positive constant that lies in the region of convergence of the integrand and

$$\Phi_{f_i}(s) \triangleq \mathcal{E}_{\gamma_{f_i}, \bar{n}_{D,i}}\{e^{-s\Delta_{f_i}(x_i, \tilde{x}_i)}\}, \quad (16)$$

$$\Phi_R(s) \triangleq \Phi_R^c(s) + \frac{1}{|\mathcal{N}(x_R)|} \sum_{\hat{x}_R \in \mathcal{N}(x_R)} \Phi_R^e(\hat{x}_R; s). \quad (17)$$

Here,  $\Phi_R^e(\hat{x}_R; s)$  and  $\Phi_R^c(s)$  are defined in Lemmas 2 and 4 in the appendix, respectively.

Asymptotic expressions for the MGFs  $\Phi_{f_i}(s)$ ,  $\Phi_R^e(\hat{x}_R; s)$ , and  $\Phi_R^c(s)$  valid for high SNR are provided in the appendix in Lemmas 1, 2, and 4, respectively. With these asymptotic expressions for the MGFs at hand, an asymptotic result for the PEP  $P(\mathbf{x} \rightarrow \tilde{\mathbf{x}})$  can be calculated based on (15). However, we postpone the derivation of the asymptotic PEP until Subsection III-B, since the computation of the asymptotic PEP depends on the actual values of  $\mathbf{x}$  and  $\tilde{\mathbf{x}}$ , which in turn depend on the considered signal constellation  $\mathcal{X}$ .

For derivation of the SER and BER, the following proposition is useful (please refer to [8] for a detailed proof).

**Proposition 1:** Assume without loss of generality that  $\bar{\gamma}_{f_i} = \zeta_{f_i} \bar{\gamma}$ ,  $\bar{\gamma}_{g_i} = \zeta_{g_i} \bar{\gamma}$ ,  $1 \leq i \leq N_s$ , and  $\bar{\gamma}_R = \zeta_R \bar{\gamma}$ , where  $\zeta_{f_i}$ ,  $\zeta_{g_i}$  and  $\zeta_R$  are finite (positive) constants, which are independent of  $\bar{\gamma}$ , and define the diversity gain associated with the PEP as  $G_{d,\text{PEP}} \triangleq -\lim_{\bar{\gamma} \rightarrow \infty} \log(P(\mathbf{x} \rightarrow \tilde{\mathbf{x}})) / \log(\bar{\gamma})$ . The diversity gain is then given by  $G_{d,\text{PEP}} = d_H(\mathbf{x}, \tilde{\mathbf{x}})$ , where  $d_H(\mathbf{x}, \tilde{\mathbf{x}})$  denotes the Hamming distance between data vector  $\mathbf{s}_e \triangleq [s_1, \dots, s_{N_s}, s_R]^T$  and  $\tilde{\mathbf{s}}_e \triangleq [\tilde{s}_1, \dots, \tilde{s}_{N_s}, \tilde{s}_R]^T$ . Furthermore, for all possible pairs  $(\mathbf{x}, \tilde{\mathbf{x}})$  we have  $d_H(\mathbf{x}, \tilde{\mathbf{x}}) \geq 2$ .

#### B. Asymptotic SER and BER

We use a truncated union-bound, where we include only nearest neighbor error events, to obtain an asymptotic expression for the SER based on the asymptotic PEP  $P(\mathbf{x} \rightarrow \tilde{\mathbf{x}})$ . In particular, a highly accurate approximation for the asymptotic SER of the  $i$ th source,  $P_s^i$ , is given by

$$P_s^i \triangleq \frac{1}{M^{N_s}} \sum_{\mathbf{x} \in \mathcal{X}^{N_s}} \sum_{\tilde{\mathbf{x}} \in \mathcal{C}_i(\mathbf{x})} P(\mathbf{x} \rightarrow \tilde{\mathbf{x}}), \quad (18)$$

where

$$\mathcal{C}_i(\mathbf{x}) \triangleq \{\tilde{\mathbf{x}} | \tilde{x}_j \in \mathcal{N}(x_j) \cup \{x_j\}, j \neq i, \tilde{x}_j \in \mathcal{N}(x_j), j = i, d_H(\mathbf{x}, \tilde{\mathbf{x}}) = 2\}. \quad (19)$$

In (19), we have only included error events with  $d_H(\mathbf{x}, \tilde{\mathbf{x}}) = 2$  since error events with  $d_H(\mathbf{x}, \tilde{\mathbf{x}}) > 2$  yield a higher diversity gain (cf. Proposition 1) and thus, their contribution to the asymptotic SER is negligible.



TABLE I

COEFFICIENTS  $C_{\mathcal{X}}^1$  AND  $C_{\mathcal{X}}^2$  FOR DIFFERENT SIGNAL CONSTELLATIONS.FOR  $M$ -PSK AND  $M$ -QAM WE HAD  $d = 2 \sin(\frac{\pi}{M})$ , $\nu \triangleq 4(\sin(\frac{2\pi}{M})^2 - \sin(\frac{\pi}{M})^2)$  AND  $d = \sqrt{\frac{6}{M-1}}$ , RESPECTIVELY.

$\mathcal{X}$	$C_{\mathcal{X}}^1$	$C_{\mathcal{X}}^2$
BPSK	$\frac{45+\sqrt{5}}{160}$	$\frac{3}{16}$
$M$ -PSK	$\frac{8+\frac{2}{\sqrt{5}}}{d^4} + \frac{1-\sqrt{\frac{d^2}{d^2+4\nu}}}{d^2\nu}$	$\frac{6}{d^4}$
$M$ -QAM	$\frac{18.27-\frac{17.44}{\sqrt{M}}-\frac{0.83}{\sqrt{M^3}}}{d^4}$	$\frac{12}{d^4} \left(1 - \frac{1}{\sqrt{M}}\right)$

We are now ready to state our main result. In particular, in the following proposition we use (18) to obtain the asymptotic SER for BPSK,  $M$ -PSK, and  $M$ -QAM signal constellations.

**Proposition 2:** For an NCCD system with  $N_s$  sources the asymptotic SER for the  $i$ th source is given by

$$P_{s,\mathcal{X}}^i \triangleq \frac{1}{\bar{\gamma}_{f_i}} \left( C_{\mathcal{X}}^1 \sum_{i=1}^{N_s} \frac{1}{\bar{\gamma}_{g_i}} + C_{\mathcal{X}}^2 \left[ \sum_{\substack{j=1 \\ j \neq i}}^{N_s} \frac{1}{\bar{\gamma}_{f_j}} + \frac{1}{\bar{\gamma}_R} \right] \right), \quad (20)$$

where  $\mathcal{X} \in \{\text{BPSK}, M\text{-PSK}, M\text{-QAM}\}$ , and  $C_{\mathcal{X}}^1$  and  $C_{\mathcal{X}}^2$  are tabulated in Table I.

*Proof:* The asymptotic SER  $P_{s,\mathcal{X}}^i$  can be calculated by using (18) and (19) along with (15). Because of space limitations, we limit the proof to the BPSK case, i.e.,  $\mathcal{X} = \{\pm 1\}$ . However, a similar approach can be used to obtain the SER for  $M$ -PSK,  $M$ -QAM, and any other signal constellation. In the BPSK case, for a given transmit signal vector  $\mathbf{x}$ , the set  $\mathcal{C}_i(\mathbf{x})$  in (19) contains  $N_s$  elements, i.e.,  $\mathcal{C}_i(\mathbf{x}) = \{\tilde{\mathbf{x}}^1, \dots, \tilde{\mathbf{x}}^{N_s}\}$ ,  $\tilde{\mathbf{x}}^l \triangleq [\tilde{x}_1^l, \dots, \tilde{x}_{N_s}^l]^T$ , where

$$\tilde{x}_j^l = \begin{cases} -x_j, & j = i, j = l \\ x_j, & \text{otherwise} \end{cases}, \quad 1 \leq l \leq N_s, 1 \leq j \leq N_s. \quad (21)$$

In the following, we first obtain the asymptotic PEP  $P(\mathbf{x} \rightarrow \tilde{\mathbf{x}}^l)$  for the case  $l = i$  before we consider the case  $l \neq i$ .

**Case 1 ( $l = i$ ):** Defining  $d_j \triangleq |x_j - \tilde{x}_j^l|$ , we have  $d_j = 2$ ,  $j = i$ , and  $d_j = 0$ ,  $j \neq i$ . As a result, from Lemma 1 we obtain  $\Phi_{f_j}(s) \triangleq \frac{1}{4s(1-s)\bar{\gamma}_{f_j}}$ ,  $j = i$  and  $\Phi_{f_j}(s) \triangleq 1$ ,  $j \neq i$ . Furthermore, taking into account that for  $l = i$  we have  $\tilde{x}_R = -x_R$ , based on Lemmas 2 and 4, (15), and (17) we obtain

$$P(\mathbf{x} \rightarrow \tilde{\mathbf{x}}^i) = \frac{1}{16\pi^2 j \bar{\gamma}_{f_i}} \int_0^{\pi/2} \int_{c-j\infty}^{c+j\infty} \frac{1}{s(1-s)} \times \left( \frac{1}{\bar{\gamma}_{\text{eq}} s (1 - 16 \sin^4 \theta s^2)} + \frac{1}{\bar{\gamma}_R s (1-s)} \right) \frac{ds}{s} d\theta, \quad (22)$$

where we have used that for BPSK  $\mathcal{N}(x_R) = \{-x_R\}$ ,  $\alpha = \beta = 1$ ,  $\bar{d}_R(\hat{x}_R) = -4$ , and  $d_R = 2$  are valid.

The inner complex integral in (22) can be calculated using the standard inverse Laplace transform techniques such as partial fraction expansion. This yields

$$P(\mathbf{x} \rightarrow \tilde{\mathbf{x}}^i) \triangleq \frac{1}{\bar{\gamma}_{f_i}} \left( C_{\text{BPSK}}^1 \sum_{i=1}^{N_s} \frac{1}{\bar{\gamma}_{g_i}} + C_{\text{BPSK}}^2 \frac{1}{\bar{\gamma}_R} \right), \quad (23)$$

where  $C_{\text{BPSK}}^1$  and  $C_{\text{BPSK}}^2$  are given in Table I.

**Case 2 ( $l \neq i$ ):** For  $l \neq i$ , from Lemma 1 we have  $\Phi_{f_j}(s) \triangleq \frac{1}{4s(1-s)\bar{\gamma}_{f_j}}$ ,  $j = i, j = l$ , and  $\Phi_{f_j}(s) \triangleq 1$ , otherwise.

Furthermore, in this case  $\tilde{x}_R = x_R$  is valid and therefore based on Lemmas 2 and 4 and (17) we have  $\Phi_R(s) \triangleq 1$ . Therefore, using (15) we arrive at

$$P(\mathbf{x} \rightarrow \tilde{\mathbf{x}}^l) = \frac{1}{32\pi j \bar{\gamma}_{f_i} \bar{\gamma}_{f_l}} \int_{c-j\infty}^{c+j\infty} \frac{1}{s^3(1-s)^2} ds = \frac{C_{\text{BPSK}}^2}{\bar{\gamma}_{f_i} \bar{\gamma}_{f_l}} \quad (24)$$

for  $l \neq i$ ,  $1 \leq l \leq N_s$ .

Finally, combining (18), (23), and (24) yields (20) for BPSK. ■

**Remark 2:** For  $N_s = 1$  the considered NCCD system reduces to a CD system with a single decode-and-forward relay and C-MRC at the destination [7]. Letting  $N_s = 1$  in (20), the asymptotic SER for this system can therefore be obtained as

$$P_{s,\mathcal{X}} \triangleq \frac{1}{\bar{\gamma}_{f_i}} \left( \frac{C_{\mathcal{X}}^1}{\bar{\gamma}_{g_i}} + \frac{C_{\mathcal{X}}^2}{\bar{\gamma}_R} \right), \quad (25)$$

which is a new result since the analysis presented in [7] reveals only the diversity gain but does not provide a tight approximation for the asymptotic SER.

We note that having obtained the asymptotic SER, for Gray labeling, the asymptotic BER of the  $i$ th source,  $P_{b,\mathcal{X}}^i$ , can be tightly approximated as

$$P_{b,\mathcal{X}}^i \triangleq \frac{1}{\log_2(M)} P_{s,\mathcal{X}}^i. \quad (26)$$

### C. Diversity Gain and Coding Gain

Letting  $\bar{\gamma}_{f_i} = \zeta_{f_i} \bar{\gamma}$ ,  $\bar{\gamma}_{g_i} = \zeta_{g_i} \bar{\gamma}$ ,  $1 \leq i \leq N_s$ , and  $\bar{\gamma}_R = \zeta_R \bar{\gamma}$ , where  $\zeta_{f_i}$ ,  $\zeta_{g_i}$  and  $\zeta_R$  are finite (positive) constants, we can express the asymptotic SER of the  $i$ th source as  $P_{s,\mathcal{X}}^i \triangleq (G_{c,\text{SER}}^i \bar{\gamma})^{-G_{d,\text{SER}}^i}$ , where  $G_{d,\text{SER}}^i$  and  $G_{c,\text{SER}}^i$  are the diversity gain and the coding gain corresponding to the asymptotic SER, respectively. Thus,  $G_{d,\text{SER}}^i$  and  $G_{c,\text{SER}}^i$  correspond to the negative asymptotic slope and a relative horizontal shift of the SER curve when plotted as a function of  $\bar{\gamma}$  on a double-logarithmic scale, respectively. Based on (20) we obtain  $G_{d,\text{SER}}^i = 2$  and

$$G_{c,\text{SER}}^i [\text{dB}] = 5 \log_{10}(\zeta_{f_i}) - 5 \log_{10} \left( C_{\mathcal{X}}^1 \sum_{i=1}^{N_s} \zeta_{g_i}^{-1} + C_{\mathcal{X}}^2 \left[ \sum_{\substack{j=1 \\ j \neq i}}^{N_s} \zeta_{f_j}^{-1} + \zeta_R^{-1} \right] \right). \quad (27)$$

**Remark 3:** A diversity gain of  $G_{d,\text{SER}}^i = 2$  is achieved by all sources irrespective of the number of sources  $N_s$ . However, from (27) it is evident that the coding gain is dependent on the number of sources, the signal constellation  $\mathcal{X}$ , and the relative link qualities  $\zeta_{f_i}$ ,  $\zeta_{g_i}$ , and  $\zeta_R$ .

## IV. RESULTS AND SYSTEM OPTIMIZATION

In this section, we verify the analytical results derived in Section III with computer simulations and exploit these results to optimize the performance of NCCD systems.

### A. Impact of Number of Sources

In Fig. 2, we show the average BER (average of the BERs of all sources) of an NCCD system for different numbers of sources as a function of  $\bar{\gamma}$  for BPSK.  $\zeta_{f_i} = \zeta_{g_i} = 1$ ,  $1 \leq i \leq N_s$ ,  $\zeta_R = 1$  are assumed, i.e., all links in the network

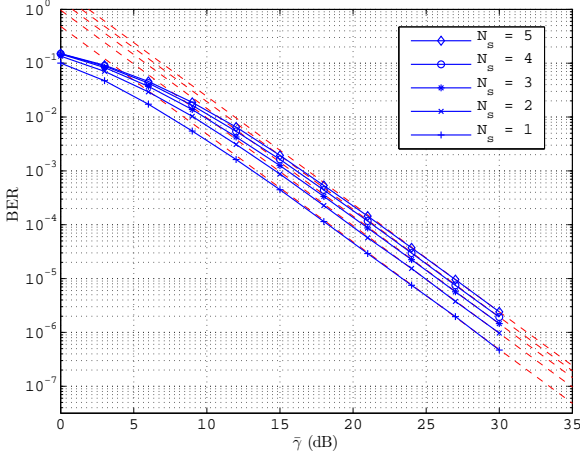


Fig. 2. Average BER vs. SNR  $\bar{\gamma}$  of an NCCD system with  $N_s$  sources and BPSK. Solid lines with markers: Simulated BER. Dashed lines: Asymptotic BER [(20), (26)].

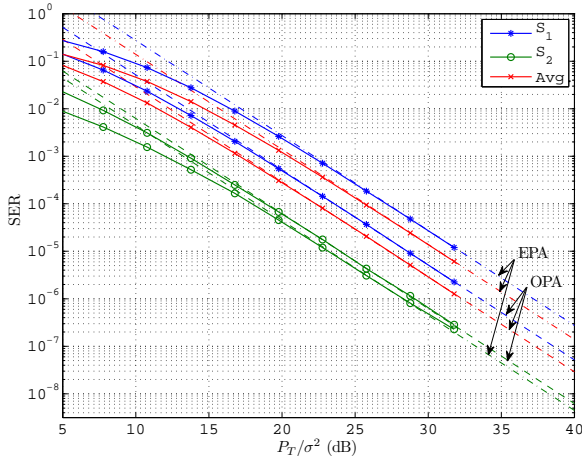


Fig. 3. SER vs.  $P_T/\sigma^2$  of an NCCD system with 8-PSK for OPA and EPA. Solid lines with markers: Simulated SER. Dashed lines: Asymptotic SER [(20)].

have the same average quality. The analytical results (dashed lines) shown in Fig. 2 were obtained with (20) and (26) and are in excellent agreement with the simulation results (solid lines with markers) for sufficiently high SNR confirming the accuracy of the approximations made in Section II and III. As expected from the analysis in Section III, a diversity gain of two is achieved irrespective of  $N_s$ . However, increasing  $N_s$  causes a horizontal shift of the asymptotic BER and a performance degradation.

### B. Performance Optimization

Similar to the case of conventional CD in [9], the asymptotic SER expression in (20) may be used for optimization of the NCCD system including optimal power allocation, relay selection, and relay placement. Because of space constraints, we only briefly discuss the power optimization problem here. Considering (20) and the definition of the SNRs  $\bar{\gamma}_{g_i}$ ,  $\bar{\gamma}_{f_i}$ ,  $1 \leq i \leq N_s$ , and  $\bar{\gamma}_R$ , it is obvious that the average SER of all sources is a posynomial in the transmit powers  $P_i$ ,  $1 \leq i \leq N_s$ , and  $P_R$  [10]. Thus, the problem of optimizing the transmit powers for minimization of the average SER under a joint transmit power constraint  $\sum_{i=1}^{N_s} P_i + P_R \leq P_T$  ( $P_T$ : maximum transmit power) can be cast into a geometric program (GP) [10] as was done for conventional CD systems

in [9]. Thus, the optimal power allocation (OPA) problem can be efficiently solved using standard tools [10].

Results for OPA as a function of  $P_T/\sigma^2$  are shown in Fig. 3 for an NCCD system with 8-PSK,  $N_s = 2$ ,  $\Omega_{f_1} = \Omega_{g_1} = 1$ ,  $\Omega_{f_2} = \Omega_{g_2} = 100$ ,  $\Omega_R = 100$ , and  $\sigma_{n_{D,i}}^2 = \sigma_{n_{R,i}}^2 = \sigma_{n_{D,R}}^2 \triangleq \sigma^2$ . Specifically, we show in Fig. 3 the SERs of both sources  $S_i$ ,  $i \in \{1, 2\}$  and the average SER of both sources, and compare OPA with equal power allocation (EPA), where  $P_1 = P_2 = P_R = P_T/3$ . Fig. 3 shows that OPA improves the average asymptotic SER (i.e., the cost function for optimization) by 3.4 dB compared to EPA. The individual SERs of  $S_1$  and  $S_2$  reveal that OPA improves the SER of  $S_1$ , which has the weaker channel, at the expense of a small degradation of the SER of source  $S_2$  by allocating more power to  $S_1$  than to  $S_2$  (and the relay).

## V. CONCLUSIONS

In this paper, we studied NCCD with general  $M$ -ary modulation and proposed a simple C-MRC diversity combining scheme which achieves the maximum diversity of the considered system even if erroneous decisions at the relay are taken into account. Assuming independent Rayleigh fading for all links in the network, we derived closed-form expressions for the asymptotic SER and BER of the considered NCCD system. These simple and elegant expressions provide insight into the impact of various system and channel parameters on performance and can be exploited for performance optimization and system design. Simulation results confirmed the accuracy of the presented asymptotic SER and BER results and revealed that optimal power allocation can improve performance by several decibels.

## APPENDIX

In this appendix, we provide four lemmas. Due to space limitations, we present only sketches of the proofs.

**Lemma 1:** The asymptotic behavior of  $\Phi_{f_i}(s)$ ,  $1 \leq i \leq N_s$ , for  $\bar{\gamma}_{f_i} \rightarrow \infty$  is given by

$$\Phi_{f_i}(s) \triangleq \frac{1}{d_i^2 s (1-s) \bar{\gamma}_{f_i}}, \quad (28)$$

for  $d_i \triangleq |x_i - \tilde{x}_i| \neq 0$  and  $\Phi_{f_i}(s) \triangleq 1$  for  $d_i = 0$ .

*Proof:* For  $d_i \neq 0$ , following the same steps as in [11, Section IV.A] for  $I_f(s) \triangleq \mathcal{E}_{\gamma_{f_i}} \{e^{-s \Delta_{f_i}(x_i, \tilde{x}_i)}\}$  we obtain

$$I_f(s) = \frac{1}{d_i^2 s \bar{\gamma}_{f_i}} \sum_{i=0}^{\infty} \frac{1}{i!} s^i |\bar{n}_{D,i}|^{2i} = \frac{e^{|\bar{n}_{D,i}|^2 s}}{d_i^2 s \bar{\gamma}_{f_i}}. \quad (29)$$

In particular, (29) can be obtained from [11, Eq. (14)] by adjusting the notation of [11] to the problem at hand.  $\Phi_{f_i}(s) = \mathcal{E}_{\bar{n}_{D,i}} \{I_f(s)\}$  can then be calculated by averaging  $I_f(s)$  with respect to the Rayleigh distributed RV  $|\bar{n}_{D,i}|^2$  leading to the result in (28). For  $d_i = 0$ ,  $\Phi_{f_i}(s) \triangleq 1$  follows from the definition of  $\Phi_{f_i}(s)$ . ■

**Lemma 2:** The asymptotic behavior of  $\Phi_R^e(\hat{x}_R; s) = \mathcal{E}_{\gamma_{eq}, \gamma_R, \bar{n}_{D,R}} \{\beta Q(\sqrt{2\alpha \gamma_{eq}}) e^{-s \lambda_R \Delta_R(x_R, \hat{x}_R, \hat{x}_R)}\}$  for  $\bar{\gamma}_{g_i} \rightarrow \infty$ ,  $1 \leq i \leq N_s$ , and  $\bar{\gamma}_R \rightarrow \infty$  is given by

$$\Phi_R^e(\hat{x}_R; s) \triangleq \frac{1}{\pi} \int_0^{\pi/2} \frac{\beta}{\bar{\gamma}_{eq}(\bar{d}_R(\hat{x}_R)s + \frac{\alpha}{\sin^2 \theta})} d\theta. \quad (30)$$

where  $\bar{d}_R(\hat{x}_R) \triangleq |\tilde{x}_R - \hat{x}_R|^2 - |x_R - \hat{x}_R|^2$ .

*Proof:* We start the proof by using the alternative representation of the  $Q$ -function,  $Q(x) = \frac{1}{\pi} \int_0^{\pi/2} e^{-x^2/\sin^2 \theta} d\theta$ , to express  $\Phi_R^e(\hat{x}_R; s)$  as

$$\Phi_R^e(\hat{x}_R; s) = \frac{\beta}{\pi} \int_0^{\pi/2} \mathcal{E}_{\bar{n}_{D,R}} \{ \Phi(s, \theta) \} d\theta, \quad (31)$$

where

$$\Phi(s, \theta) \triangleq \mathcal{E}_{\gamma_{\text{eq}}, \gamma_R} \left\{ e^{-\frac{\alpha \gamma_{\text{eq}}}{\sin^2 \theta}} e^{-s \lambda_R \Delta_R(x_R, \tilde{x}_R, \hat{x}_R)} \right\}. \quad (32)$$

Using the Taylor series expansion  $e^x = \sum_{i=0}^{\infty} x^i/i!$  along with (9) and (14) in (32) leads to

$$\Phi(s, \theta) = \sum_{i=0}^{\infty} \frac{2^i \xi_i}{(2i)!} |\bar{n}_{D,R}|^{2i} s^{2i} \Psi_i(s, \theta), \quad (33)$$

where  $\xi_i \triangleq \frac{\Gamma(i+1/2)}{\sqrt{\pi} \Gamma(i+1)}$  and  $\Psi_i(s, \theta) = \Psi_i^1(s, \theta) + \Psi_i^2(s, \theta)$  with

$$\begin{aligned} \Psi_i^1(s, \theta) &\triangleq \frac{d_R^{2i}}{\bar{\gamma}_{\text{eq}} \bar{\gamma}_R} \int_0^{\gamma_{\text{eq}}} d\gamma_{\text{eq}} e^{-\left(\frac{\alpha \gamma_{\text{eq}}}{\sin^2 \theta} + \gamma_{\text{eq}}/\bar{\gamma}_{\text{eq}}\right)} \\ &\quad \times \int_0^{\gamma_{\text{eq}}} d\gamma_R \gamma_R^i e^{-(\gamma_R \bar{d}_R(\hat{x}_R)s + \gamma_R/\bar{\gamma}_R)}, \end{aligned} \quad (34)$$

$$\begin{aligned} \Psi_i^2(s, \theta) &\triangleq \frac{d_R^{2i}}{\bar{\gamma}_{\text{eq}} \bar{\gamma}_R} \int_0^{\gamma_{\text{eq}}} d\gamma_{\text{eq}} \gamma_{\text{eq}}^{2i} e^{-\gamma_{\text{eq}}(\bar{d}_R(\hat{x}_R) + \frac{\alpha}{\sin^2 \theta} + 1/\bar{\gamma}_{\text{eq}})} \\ &\quad \times \int_{\gamma_{\text{eq}}}^{\infty} d\gamma_R \gamma_R^{-i} e^{-(\gamma_R/\bar{\gamma}_R)}. \end{aligned} \quad (35)$$

In the following, we find the asymptotic behavior of  $\Psi_i^1(s, \theta)$  and  $\Psi_i^2(s, \theta)$  for  $\bar{\gamma}_{\text{eq}}, \bar{\gamma}_R \rightarrow \infty$ , respectively. We first write (34) as

$$\begin{aligned} \Psi_i^1(s, \theta) &= \frac{d_R^{2i}}{\bar{\gamma}_{\text{eq}} \bar{\gamma}_R} \int_0^{\gamma_{\text{eq}}} e^{-\left(\frac{\alpha \gamma_{\text{eq}}}{\sin^2 \theta} + \gamma_{\text{eq}}/\bar{\gamma}_{\text{eq}}\right)} \left[ \frac{i!}{(\bar{d}_R(\hat{x}_R)s + 1/\bar{\gamma}_R)^{i+1}} \right. \\ &\quad \left. - \sum_{k=0}^i \frac{i! \gamma_{\text{eq}}^k e^{(\bar{d}_R(\hat{x}_R)s + 1/\bar{\gamma}_R)\gamma_{\text{eq}}}}{k! (\bar{d}_R(\hat{x}_R)s + 1/\bar{\gamma}_R)^{i-k+1}} \right] d\gamma_{\text{eq}} \\ &\triangleq o(\bar{\gamma}_{\text{eq}}^{-1} \bar{\gamma}_R^{-1}). \end{aligned} \quad (36)$$

Next, we rewrite (35) as

$$\Psi_i^2(s, \theta) = \frac{d_R^{2i}}{\bar{\gamma}_{\text{eq}} \bar{\gamma}_R} \int_0^{\gamma_{\text{eq}}} \gamma_{\text{eq}}^{2i} e^{-\gamma_{\text{eq}} A(s, \theta)} \Gamma(1-i, \gamma_{\text{eq}}/\bar{\gamma}_R) d\gamma_{\text{eq}}, \quad (37)$$

where  $A(s, \theta) \triangleq \bar{d}_R(\hat{x}_R)s + \frac{\alpha}{\sin^2 \theta} + 1/\bar{\gamma}_{\text{eq}}$  and  $\Gamma(\cdot, \cdot)$  is the incomplete Gamma function. Based on (37) and the asymptotic properties of  $\Gamma(\cdot, z)$  for  $z \rightarrow 0$ , we obtain

$$\Psi_i^2(s, \theta) \triangleq \begin{cases} o(\bar{\gamma}_{\text{eq}}^{-1} \bar{\gamma}_R^{-1}) & i > 1 \\ \frac{2 \log(\bar{\gamma}_R)}{(A(s, \theta))^3 \bar{\gamma}_{\text{eq}} \bar{\gamma}_R} & i = 1 \\ \frac{1}{\bar{\gamma}_{\text{eq}} (\bar{d}_R(\hat{x}_R)s + \frac{\alpha}{\sin^2 \theta})} & i = 0 \end{cases} \quad (38)$$

From (37) and (38), we therefore obtain  $\Psi_i(s, \theta) = \Psi_i^1(s, \theta) + \Psi_i^2(s, \theta) \triangleq \Psi_i^2(s, \theta)$ . Substituting this result into (33) leads to (30) upon using (31). ■

**Lemma 3:** The asymptotic behavior of  $I(s) \triangleq \mathcal{E}_{\gamma_{\text{eq}}, \gamma_R, \bar{n}_{D,R}} \{ e^{-s \lambda_R \Delta_R(x_R, \tilde{x}_R, x_R)} \}$  for  $\bar{\gamma}_{g_i} \rightarrow \infty$ ,  $1 \leq i \leq N_s$ ,  $\bar{\gamma}_R \rightarrow \infty$  is given by

$$I(s) \triangleq \frac{1}{\bar{\gamma}_{\text{eq}} d_R^2 s} - \frac{1}{\bar{\gamma}_R d_R^2 s(s-1)}, \quad (39)$$

for  $d_R \triangleq |\tilde{x}_R - x_R| \neq 0$ , while  $I(s) = 1$  is valid for  $d_R = 0$ .

*Proof:* Since  $\lambda_R \Delta_R(x_R, \tilde{x}_R, x_R) = \gamma_m d_R^2 + \frac{2\gamma_m}{\sqrt{\gamma_R}} d_R \Re\{\bar{n}_{D,R}^*\}$  with  $\gamma_m \triangleq \min\{\gamma_{\text{eq}}, \gamma_R\}$ , we conclude that  $I(s) = 1$  is valid for  $d_R = 0$ . For  $d_R \neq 0$  using a similar approach as in the proof of Lemma 2, we obtain

$$I(s|\bar{n}_{D,R}) = \mathcal{E}_{\bar{n}_{D,R}} \left\{ \sum_{i=0}^{\infty} \frac{2^i \xi_i}{(2i)!} |\bar{n}_{D,R}|^{2i} s^{2i} \Upsilon_i(s, \theta) \right\}, \quad (40)$$

where  $\Upsilon_i(s, \theta) = \Upsilon_i^1(s, \theta) + \Upsilon_i^2(s, \theta)$  with

$$\Upsilon_i^1(s, \theta) \triangleq \frac{i!}{\bar{\gamma}_R d_R^2 s^{i+1}}, \quad \Upsilon_i^2(s, \theta) \triangleq \begin{cases} o(\bar{\gamma}_{\text{eq}}^{-1} \bar{\gamma}_R^{-1}) & i > 1 \\ \frac{2 \log(\bar{\gamma}_R)}{d_R^6 s^3 \bar{\gamma}_{\text{eq}} \bar{\gamma}_R} & i = 1 \\ \frac{1}{\bar{\gamma}_{\text{eq}} d_R^2 s} & i = 0 \end{cases} \quad (41)$$

We therefore arrive at

$$\Upsilon_i(s, \theta) \triangleq \begin{cases} \frac{i!}{\bar{\gamma}_R d_R^2 s^{i+1}} & i \geq 1 \\ \frac{1}{d_R^2 s} \left( \frac{1}{\bar{\gamma}_{\text{eq}}} + \frac{1}{\bar{\gamma}_R} \right) & i = 0 \end{cases} \quad (42)$$

Substituting (42) into (40) results in (39) upon averaging (40) over the Rayleigh distributed RV  $|\bar{n}_{D,R}|$ . ■

**Lemma 4:** The asymptotic behavior of  $\Phi_R^c(s) \triangleq \mathcal{E}_{\gamma_{\text{eq}}, \gamma_R, \bar{n}_{D,R}} \{ [1 - \beta Q(\sqrt{2\alpha \gamma_{\text{eq}}})] e^{-s \lambda_R \Delta_R(x_R, \tilde{x}_R, x_R)} \}$  for  $\bar{\gamma}_{g_i} \rightarrow \infty$ ,  $1 \leq i \leq N_s$ ,  $\bar{\gamma}_R \rightarrow \infty$  is given by

$$\begin{aligned} \Phi_R^c(s) &\triangleq \frac{1}{\pi} \int_0^{\pi/2} \left( \frac{2}{\bar{\gamma}_{\text{eq}} d_R^2 s} - \frac{2}{\bar{\gamma}_R d_R^2 s(s-1)} \right. \\ &\quad \left. - \frac{\beta}{\bar{\gamma}_{\text{eq}} d_R^2 (s + \frac{\alpha}{\sin^2 \theta d_R^2})} \right) d\theta, \end{aligned} \quad (43)$$

for  $d_R \neq 0$ , while  $\Phi_R^c(s) \triangleq 1$  is valid for  $d_R = 0$ .

*Proof:* We first note that  $\Phi_R^c(s) = I(s) - \Phi_R^e(x_R; s)$ . For  $d_R \neq 0$  combining (30) and (39) readily results in (43). For  $d_R = 0$  from (30) and (39) we obtain  $\Phi_R^c(s) = 1 - \frac{1}{\pi \bar{\gamma}_{\text{eq}}} \int_0^{\pi/2} \frac{\beta \sin^2 \theta}{\alpha} d\theta \triangleq 1$ . ■

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